Preferential attachment without vertex growth: emergence of the giant component

Lutz Warnke

Georgia Tech

Joint work with Svante Janson

Model studied in this talk

Preferential attachment variant of the Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step add one new edge:
  edge vw added with probability proportional to $(d_v + \alpha) \cdot (d_w + \alpha)$

Remarks

- $\alpha > 0$ is fixed parameter, d_v is current degree of vertex v
- *'Rich-get-richer' Preferential attachment mechanism*:
  vertices with higher degree more likely to be joined
- *Hybrid model*: fixed vertex-set (ER) and preferential attachment (PA)
- *Limiting case* $\alpha = \infty$: recovers uniform Erdős–Rényi process

This talk: Answer question of Pittel (2010, Adv. Math)

Study 'giant component' phase transition in this hybrid model

Previous work on this hybrid Model

Preferential attachment variant of the Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step add one new edge:
  edge vw added with probability proportional to $(d_v + \alpha) \cdot (d_w + \alpha)$

Remarks

- $\alpha > 0$ is fixed parameter, d_v is current degree of vertex v
- *'Rich-get-richer' Preferential attachment mechanism*:
  vertices with higher degree more likely to be joined
- *Hybrid model*: fixed vertex-set (ER) and preferential attachment (PA)
- *Limiting case* $\alpha = \infty$: recovers uniform Erdős–Rényi process

Previous work

- Degree distribution (Ben-Naim–Krapisvky, Samalam)
- Graph-Limits (Borgs–Chayes–Lovász–Sós-Vesztergombi, Rath–Szakács)
- Giant component phase transition (Pittel, Ben-Naim–Krapivsky)

Motivation to study this hybrid Model

Preferential attachment variant of the Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step add one new edge:
  edge vw added with probability proportional to $(d_v + \alpha) \cdot (d_w + \alpha)$

Remarks

- $\alpha > 0$ is fixed parameter, d_v is current degree of vertex v
- *'Rich-get-richer' Preferential attachment mechanism*:
  vertices with higher degree more likely to be joined
- *Hybrid model*: fixed vertex-set (ER) and preferential attachment (PA)
- *Limiting case* $\alpha = \infty$: recovers uniform Erdős–Rényi process

Motivation

- *Natural model*: suggested in physics, combinatorics, probability
- *Universality*: similarities/differences to reference models
- Phase transition details an open problem since 2010 (Pittel)

Phase Transition in hybrid Model: prev. work

Preferential attachment variant of the Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step add one new edge:

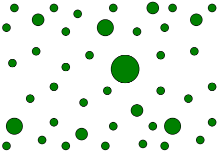
  edge vw added with probability proportional to $(d_v + \alpha) \cdot (d_w + \alpha)$

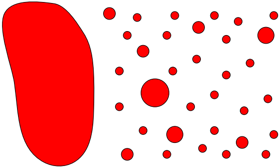
Phase transition (Pittel, 2010, Adv. Math)

Size of largest component 'dramatically changes' after $\approx t_c n$ steps. Whp

$$L_1(tn) = \begin{cases} O(\log n) & \text{if } t < t_c \\ \Theta(n) & \text{if } t > t_c \end{cases}$$

where the critical time is $t_c = \frac{\alpha}{2(\alpha+1)}$.





Phase Transition in hybrid Model: prev. work

Preferential attachment variant of the Erdős–Rényi random graph process

- Start with an empty graph on n vertices
- In each step add one new edge:

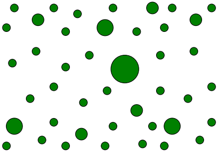
  edge vw added with probability proportional to $(d_v + \alpha) \cdot (d_w + \alpha)$

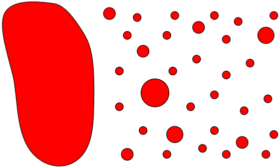
Phase transition (Pittel, 2010, Adv. Math)

Size of largest component 'dramatically changes' after $\approx t_c n$ *steps.* Whp

$$L_1(tn) = \begin{cases} \Theta(\varepsilon^{-2} \log(\varepsilon^3 n)) & \text{if } t = t_c - \varepsilon \text{ and } \varepsilon \gg n^{-1/3} \\ \Theta(\varepsilon n) & \text{if } t = t_c + \varepsilon \text{ and } \varepsilon \gg n^{-1/4} \end{cases}$$

where the critical time is $t_c = \frac{\alpha}{2(\alpha+1)}$.





Phase Transition in hybrid Model

Linear growth of giant component (Janson–W. 2019+)

For $\varepsilon \gg n^{-1/3}$, whp the size of the largest component satisfies

$$L_1\big((1+\varepsilon)t_c n\big) \approx \frac{2\alpha}{\alpha+2}\varepsilon n$$

where the critical time is $t_c = \frac{\alpha}{2(\alpha+1)}$.

- **This solves open problem of Pittel from 2010 (Adv. Math)**
- Rigorizes statistical physics prediction for $\alpha = 1$ (Ben-Naim–Krapisvky)
- Recovers basic features of the Erdős-Rényi transition
- We can even allow for $\alpha = \alpha(n) \to a \in (0, \infty]$.
- For $\alpha \gg n^{1/3}$ it 'looks exactly' like Erdős-Rényi behaviour:

$$L_1\big((1+\varepsilon)\tfrac{n}{2}\big) \approx 2\varepsilon n$$

Glimpse of the proof

Hybrid process has *super-nice* properties

- **Degree sequence** (after m steps):
  can be understood via independent birth-processes
- **Conditioned on the degree sequence** (after m steps):
  the graph has uniform distribution, with that degree sequence

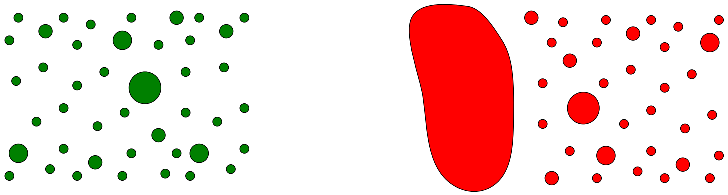
Combining these *amazing* properties

- **Can study phase transition using Configuration-Model results**
  (for graphs chosen uniformly at random according to degree sequence)

Why can we get so precise results/allow for $\alpha = \alpha(n)$?

- Because all steps are 'clean' (no delicate counting/approximations)

Summary



Phase transition in hybrid random graph process (ER+PA)

For $\varepsilon \gg n^{-1/3}$, whp the size of the largest component satisfies

$$L_1\big((1+\varepsilon)t_c n\big) \approx \tfrac{2\alpha}{\alpha+2}\varepsilon n$$

where the critical time is $t_c = \frac{\alpha}{2(\alpha+1)}$.

Remarks/Questions

- Solves problems of Pittel (2010) and Ben-Naim–Krapisvky (2012)
- Other interesting 'dynamic' random graph processes?