Prague Dimension of Random Graphs

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General "Complexity" Concepts:

- **Representation:** efficient/compact encoding of objects
- **Decomposition:** split object into smallest number of "simpler" objects
- **Dimension:** embed object into smallest number of "one-dimensional" objects

Prague Dimension: relates these concepts for Graphs

- Introduced by Nešetřil, Pultr & Rödl (1970s)
- Natural Notion: many equivalent definitions (⇝ next slide)
- Many Approaches: Algebraic, Combinatorial and Information-Theoretic
- NP-hard to determine

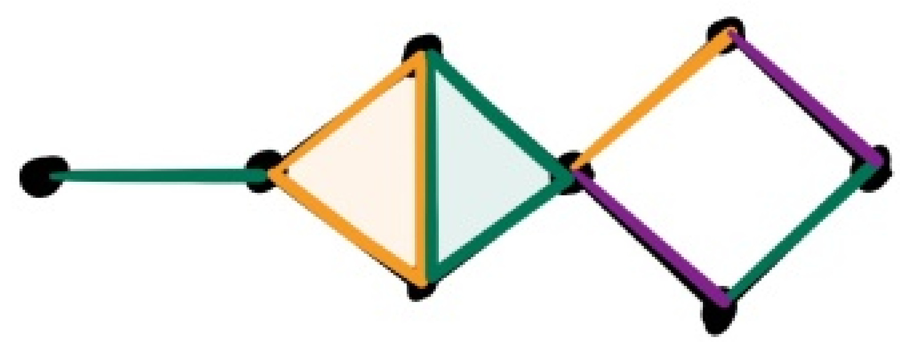
This Talk: Resolve Prague Dimension Conjecture of Füredi & Kantor

- Determine order of Prague Dimension of Random Graphs

Prague-Dimension (Nešetřil, Pultr, Rödl: 1970s)

$\dim_p(\overline{G}) :=$ min k s.t. there is clique edge-covering $\mathcal{C}$ of G which can be k-colored $= \min\limits_{\mathcal{C} \text{ of } G} \chi'(\mathcal{C}) =: cc'(G)$

(all cliques in each color-class are vx-disjoint)



$\dim_p(\overline{G}) = cc'(G) = 3$

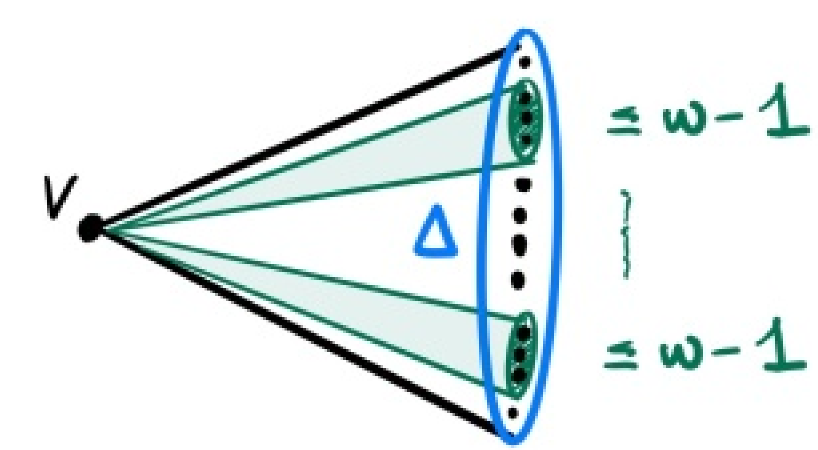
Conjecture (Füredi-Kantor)

Whp $\dim_p(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ for constant $p \in (0,1)$

- Suffices to show $cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ whp: $\dim_p(G_{n,p}) = cc'(\overline{G_{n,p}}) \stackrel{d}{=} cc'(G_{n,1-p})$

- Lower bound easy:

$$cc'(G_{n,p}) \geq \frac{\Delta(G_{n,p})}{\omega(G_{n,p})-1} \approx \frac{np}{2\log_{\frac{1}{p}} n}$$



$cc'(G) := \min\limits_{\mathcal{C} \text{ of } G} \chi'(\mathcal{C})$ = min k s.t. there is clique edge-covering $\mathcal{C}$ of G which can be k-colored (all cliques in each color-class are vx-disjoint) $=: \dim_p(\overline{G})$

Main Result (Guo, Patton, W. 2020+)

$\boxed{\text{Whp } cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right) \text{ for constant } p \in (0,1)}$

- Verifies Conjecture of Füredi-Kantor: $\dim_p(G_{n,p}) \stackrel{d}{=} cc'(G_{n,1-p}) = \Theta\left(\frac{n}{\log n}\right)$
- Extensions: $p = p(n) \to 0$ & edge-packing variant
- Difficulty: need to use/color cliques of size $O(\log n)$

Motivation / Context:

- Properties of almost all Graphs
- Covering/Decomposition Problems
- Random Greedy Paradigm

$$cc'(G) := \min_{\mathcal{C} \text{ of } G} \chi'(\mathcal{C}) = \min k \text{ s.t. there is clique edge-covering } \mathcal{C} \text{ of } G =: \mathrm{dim}_p(\overline{G})$$

which can be k-colored

(all cliques in each color-class are vx-disjoint)

Main Result (Guo, Patton, W. 2020$^+$)

$\boxed{\text{Whp } cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right) \text{ for constant } p \in (0,1)}$

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- Extensions: $p = p(n) \to 0$ & edge-packing variant
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Related Parameters/Work:

- Size: $\min_{\mathcal{C} \text{ of } G_{n,p}} |\mathcal{C}| = \Theta\left(\frac{n^2}{(\log n)^2}\right)$ whp (Frieze-Reed 1995)
- Degree: $\min_{\mathcal{C} \text{ of } G_{n,p}} \Delta(\mathcal{C}) = \Theta\left(\frac{n}{\log n}\right)$ whp (Füredi-Kantor 2018)

$cc'(G) := \min_{\mathcal{C} \text{ of } G} \chi'(\mathcal{C}) = \min k$ s.t. there is clique edge-covering $\mathcal{C}$ of G which can be k-colored (all cliques in each color-class are vx-disjoint) $=: \dim_p(\bar{G})$

Main Result (Guo, Patton, W. 2020+)

Whp $cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ for constant $p \in (0,1)$

- Verifies Conjecture of Füredi-Kantor: $\dim_p(G_{n,p}) \overset{d}{=} cc'(G_{n,1-p}) = \Theta\left(\frac{n}{\log n}\right)$
- Extensions: $p = p(n) \to 0$ & edge-packing variant
- Difficulty: need to use/color cliques of size $O(\log n)$

Proof-Strategy:

1. Find clique edge-covering: $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_I$ $\longleftarrow$ Semi-Random "nibble" Alg.
2. Color clique edge-covering: $\chi'(\mathcal{C}) \le \sum_{0 \le i \le I} \chi'(\mathcal{C}_i)$ $\longleftarrow$ Random Greedy Alg.

1 Semi-Random Edge-Decomposition into Cliques:

Construct $G_{n,p} = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_I$

$$\underbrace{\mathcal{E}_0 \quad \mathcal{E}_1 \quad \mathcal{E}_{I-1}}_{\text{By randomly deleting cliques of size } O(\log n)} \qquad \underbrace{\mathcal{E}_I}_{\text{Remaining Edges}}$$

Key-Property: G_i "looks like" G_{n,p_i} with $p_i = p \cdot e^{-i}$ (extra technical twist)

2 Random Greedy coloring of $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_I$:

$$CC'(G_{n,p}) \leq \chi'(\mathcal{E}) \leq \sum_{0 \leq i < I} \underbrace{\chi'(\mathcal{E}_i)}_{\leq O(\Delta(\mathcal{E}_i))} + \underbrace{\chi'(\mathcal{E}_I)}_{\leq 2 \cdot \Delta(G_I)}$$

DREAM $\leq O(\Delta(\mathcal{E}_i))$

by "Pippenger-Spencer" like Greedy Coloring Result

$\leq 2 \cdot \Delta(G_I)$ as $\mathcal{E}_I = E(G_I)$

$$\leq \cdots \leq O\left(\sum_{0 \leq i < I} \frac{n p_i}{\log_{\frac{1}{p_i}} n}\right) + O\left(np \cdot e^{-I}\right) \leq \cdots \leq O\left(\frac{np}{\log_{\frac{1}{p}} n}\right)$$

Random Greedy coloring of $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_I$:

$$cc'(G_{n,p}) \leq \chi'(\mathcal{C}) \leq \sum_{0 \leq i < I} \underbrace{\chi'(\mathcal{C}_i)}_{\text{DREAM } \leq O(\Delta(\mathcal{C}_i))} + \underbrace{\chi'(\mathcal{C}_I)}_{\leq 2 \cdot \Delta(G_I)} \leq \dots \leq O\left(\frac{np}{\log_{\frac{1}{p}} n}\right)$$

DREAM $\leq O(\Delta(\mathcal{C}_i))$ by "Pippenger-Spencer" like Greedy Coloring Result

$\leq 2 \cdot \Delta(G_I)$ as $\mathcal{C}_I = E(G_I)$

Technical Problem: $\mathcal{C}_i$ has cliques of size $O(\log n)$

Pippenger-Spencer (1989, "Vizing Replacement")

$\chi'(\mathcal{H}) \leq (1+\varepsilon) \cdot \Delta(\mathcal{H})$ for any $\mathcal{H}$
- k-uniform for $k = O(1)$
- approx. regular
- small codegree

Solution: Exploit that $\mathcal{C}_i$ is random set of cliques

$\to$ Can relax $k = O(1)$ to $k = O(\log n)$

$\to$ DREAM works

Chromatic Index of Random Subhypergraphs (Guo, Patton, W. 2020$^+$)

Hypergraph $\mathcal{H}$: k-uniform n vertex

- edge-uniformity: $2 \le k \le b \log n$
- approx. regular: $\deg_{\mathcal{H}}(v) = (1 \pm n^{-\sigma}) D$
- small codegree: $\mathrm{codeg}_{\mathcal{H}}(u,v) \le n^{-\sigma} D$

$\mathcal{H}_m$ = Random Subhypergraph of $\mathcal{H}$ containing $n^{1+\sigma} \le m \ll e(\mathcal{H})$ edges

Then whp $\chi'(\mathcal{H}_m) \le (1+\delta)\Delta(\mathcal{H}_m)$ for $\delta \approx \frac{b}{\sigma}$

Two Corollaries:

whp $\chi'(\mathcal{H}_m) \le \begin{cases} (1+\varepsilon)\cdot\Delta(\mathcal{H}_m) & \text{if } k = o(\log n) \rightsquigarrow \text{"Pippenger–Spencer like"} \\ O(1)\cdot\Delta(\mathcal{H}_m) & \text{if } k = O(\log n) \rightsquigarrow \text{What we apply} \end{cases}$

Algorithmic Proof:

- Natural Random Greedy Algorithm using $(1+\delta)\frac{km}{n}$ colors
- Analysis based on Differential Equation Method

Simple Random Greedy Algorithm

Possible colors: $Q := \{1, \dots, q\}$ $\quad q := (1+\delta)\frac{km}{n} \approx (1+\delta)\Delta(\mathcal{H}_m)$

For $i = 1, \dots, m$:

- sample random edge $e_i \in E(\mathcal{H})$
- color e_i with random available color from Q

Main Claim: (Under PS-like assumptions)

- Algorithm whp colors edges $e_1, \dots, e_m$

Pseudo-Random Properties: (Differential Equation Method)

Whp for all edges $e \in E(\mathcal{H})$ and steps $0 \le i \le m$:

$$Q_e(i) := \underset{\text{(after } i \text{ steps)}}{\#\text{available colors for } e} \approx \left(1 - \frac{i}{m(1+\delta)}\right)^k \cdot |Q|$$

Main Claim: (Under PS-like assumptions)

- Algorithm whp colors edges $e_1, \ldots, e_m$

Pseudo-Random Properties: (Differential Equation Method)

Whp for all edges $e \in E(\mathcal{H})$ and steps $0 \le i \le m$:

$$Q_e(i) := \begin{array}{c}\#\text{available colors for } e\\ (\text{after } i \text{ steps})\end{array} \approx \left(1 - \frac{i}{m(1+\delta)}\right)^k \cdot |Q|$$

Q: Why does uniformity $k \le b \log n$ work for $m \ge n^{1+\sigma}$ steps?

Whp algorithm never runs out of available colors:

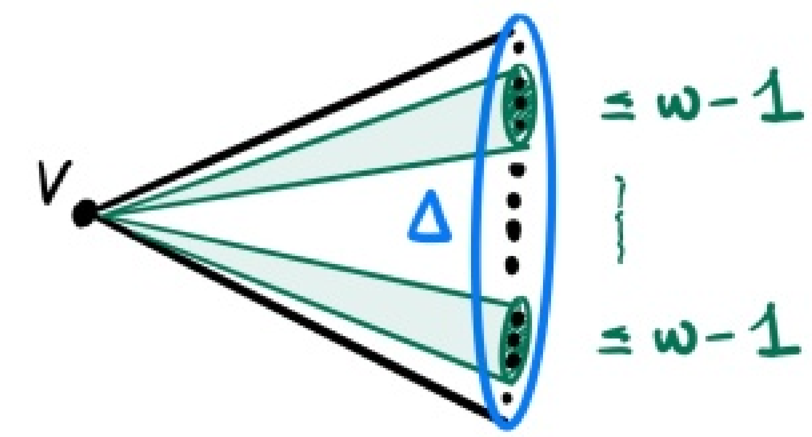
$$\min_{i \le m} Q_e(i) \ge \underbrace{\left(1 - \frac{1}{1+\delta}\right)^k}_{\ge \left(\frac{\delta}{1+\delta}\right)^{b \log n}} \cdot \underbrace{q}_{\substack{= \frac{(1+\delta)km}{n} \\ \ge \frac{m}{n} \ge n^{\sigma}}} \ge n^{-b \log(1+\frac{1}{\delta}) + \sigma} \underset{\log(1+\frac{1}{\delta}) \le \frac{1}{\delta}}{\ge} n^{-\frac{b}{\delta} + \sigma} \underset{\delta = \frac{2b}{\sigma} \text{ say}}{\gg} 1$$

Open Problem: Asymptotics of $cc'(G) = \min \chi'(\mathcal{C})$

Known Bounds (Recap):

- Whp $cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ for constant $p \in (0,1)$
- Simple lower bound:

$$cc'(G_{n,p}) \geq \frac{\Delta(G_{n,p})}{\omega(G_{n,p}) - 1} \approx \frac{np}{2\log_{\frac{1}{p}}(n)}$$



Unclear what asymptotics to guess:

- Can improve simple lower bound:

$$cc'(G_{n,p}) \geq (1-o(1)) \cdot \underbrace{(1+f(p))}_{=2 \text{ for } p=\frac{1}{2}} \cdot \frac{np}{2\log_{\frac{1}{p}}(n)} \quad \text{for some } f(p)>0.$$

Summary:

$$\dim_P(\bar{G}) := \min_{\mathcal{C} \text{ of } G} \chi'(\mathcal{C})$$

Prague Dimension of Random Graphs

Typically $\dim_P(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ for constant $p \in (0,1)$

- Verifies Conjecture of Füredi-Kantor
- Proof: 'Semi-Random' + 'Random Greedy' Alg.
- New Tool: Chrom.-Index of random subhypergr. with edge-size $O(\log n)$

Questions:

- What is asymptotics of $\dim_P(G_{n,p})$?
- What can we say about sparse case $p = p(n) \to 0$?