EXTREME LOCAL STATISTICS IN RANDOM GRAPHS:
MAXIMUM TREE EXTENSION COUNTS

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Abstract. We consider maximum rooted tree extension counts in random graphs, i.e.,
we consider \( M_n = \max_v X_v \) where \( X_v \) counts the number of copies of a given tree in \( G_{n,p} \)
rooted at vertex \( v \). We determine the asymptotics of \( M_n \) when the random graph is not
too sparse, specifically when the edge probability \( p = p(n) \) satisfies \( p(1 - p)n \gg \log n \).
The problem is more difficult in the sparser regime \( 1 \ll pn \ll \log n \), where we determine
the asymptotics of \( M_n \) for specific classes of trees. Interestingly, here our large deviation
type optimization arguments reveal that the behavior of \( M_n \) changes as we vary \( p = p(n) \),
due to different mechanisms that can make the maximum large.

1. Introduction

In extreme value theory, the distribution of the maximum \( \max_{i \in [n]} X_i \) of \( n \) many
i.i.d. random variables is a classical topic [LLR83, EKM97, BGTS04]. In this paper we
study a variant of this problem from random graph theory, where the relevant random
variables \( X_1, \ldots, X_n \) are not independent (and may also depend on \( n \)). Interestingly, for
the binomial random graph \( G_{n,p} \) we (i) discover that the qualitative behavior of the relevant
maximum changes as we vary the edge probability \( p = p(n) \), and (ii) identify the different underlying mechanisms that can make the corresponding maximum large.

The oldest extreme value problem in random graph theory concerns the maximum
vertex degree \( \Delta(G_{n,p}) \) over all \( n \) vertices of the binomial random graph \( G_{n,p} \). Indeed,
already in the 1960s it was folklore that for most edge probabilities \( p = p(n) \) of interest
the maximum degree \( \Delta \) is concentrated around the mean degree \( pn \), or more formally that

\[
\frac{\Delta}{pn} \overset{p}{\to} 1,
\]

where \( \overset{p}{\to} \) denotes convergence in probability. In the 1970s and 1980s the asymptotic
behavior of the centered random variable \( \Delta - pn \) was established for many edge proba-
bilities \( p = p(n) \) of interest, yielding in particular that, for \( p(1 - p)n \gg \log n \),

\[
\frac{\Delta - pn}{\sqrt{2p(1 - p)n \log n}} \overset{p}{\to} 1,
\]

where henceforth log stands for the natural logarithm. See [Ivc73, Bol80, Bol01] for more
details about the maximum degree \( \Delta \).

In this paper we investigate perhaps the simplest generalization of the maximum degree
that is not well understood for the binomial random graph \( G_{n,p} \). Namely, given a rooted
tree \( T \), for each vertex \( v \in [n] \) we define a local statistic \( X_v \) (deferring the precise definition
for a while) that counts ‘local’ copies of \( T \) rooted at \( v \), write \( \mu_T := \mathbb{E} X_v \) for the mean

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(which by symmetry does not depend on the choice of $v$), and then study the maximum rooted tree extension count
\[ M_n = M_{T,n} := \max_{v \in [n]} X_v \tag{3} \]
taken over all $n$ vertices of the binomial random graph $\mathbb{G}_{n,p}$. This includes the maximum degree problem as the simplest case where $T$ is an edge, rooted at one endvertex. Conceptually, the main difficulty is that $M_n$ is an extreme order statistic of $n$ random variables $X_v$ that are not independent (and their distribution depends on $n$). As we shall discuss in Sections 1.1–1.2, the main results of this paper answer the following fundamental extreme value theory questions concerning $M_n$, which are also interesting random graph theory questions in their own right:

(i) For what range of edge probabilities $p = p(n)$ does $M_n / \mu_T \xrightarrow{p} 1$ hold, i.e., is the maximum $M_n$ concentrated around the mean extension count $\mu_T$?

(ii) When $M_n / \mu_T \xrightarrow{p} 1$ holds, then what is the correct asymptotic expression for the centered random variable $M_n - \mu_T$, i.e., how much does the maximum $M_n$ typically deviate from the mean extension count $\mu_T$?

(iii) When $M_n / \mu_T \xrightarrow{p} 1$ fails, then what is the asymptotic behavior of $M_n$, i.e., can we identify a sequence $\alpha_n$ such that $M_n / \alpha_n \xrightarrow{p} 1$ holds for certain trees?

We now provide some definitions omitted so far. Let $\mathbb{G}_{n,p}$ denote the random graph with vertex set $[n] := \{1, \ldots, n\}$, in which each of the $\binom{n}{2}$ possible edges is included independently with probability $p = p(n)$; see [Gil59, ER60, Bol01, JLR00]. Given a rooted graph $H$ with root $\rho$, we consider the maximum number of copies of $H$ ‘rooted’ at a vertex $v \in [n]$, counted with multiplicity$^4$ to simplify the formulas. For each vertex $v \in [n]$ we denote by $X_v = X_{H,v}$ the number of $H$-extensions of $v$ in $\mathbb{G}_{n,p}$, i.e., $X_v$ denotes the number of injective functions $\phi : V(H) \rightarrow [n]$ such that $\phi(\rho) = v$ and $\phi(u)\phi(w) \in E(\mathbb{G}_{n,p})$ for every edge $uw \in E(H)$. Note that if $H$ has $\nu_H$ vertices and $e_H$ edges, then their expected number equals$^2$
\[ \mu_H = \mathbb{E}X_v := (n-1)_{\nu_H-1} p^{e_H} \approx n_{\nu_H-1} p^{e_H}, \]
where $e_T = v_T - 1$ for any tree $T$ (and $a_n \sim b_n$ is shorthand for $a_n / b_n \rightarrow 1$ as $n \rightarrow \infty$).

Maximum tree extension counts are a natural generalization of the maximum degree, but there is important further motivation from random graph theory. Indeed, a rooted graph $H$ is called balanced if, among all subgraphs $F \subseteq H$ containing the root vertex $\rho$, the maximizers of $e_F / (v_F - 1)$ include $H$ itself. Similarly, $H$ is called strictly balanced if $F = H$ is the only maximizer (see [JLR00, Section 3.4]). As usual for such definitions, the first step is to study the maximum extension count $M_n = M_{H,n}$ for strictly balanced $H$, which was successfully done in [Spe90, SW22]; see Section 1.3 for more details. To gain further insight into extension counts, the natural next step is to consider balanced $H$, for which even the answer to question (i) is only known for sufficiently large $p = p(n)$; see [Spe90, SW22]. In this paper we address this fundamental knowledge gap by studying questions (i) to (iii) for the ‘boundary’ class of rooted trees, which are balanced$^3$ but not strictly balanced, i.e., sit at the boundary between the known and the unknown.

1.1. How and when does $M_n$ concentrate around the mean? In this section we answer the closely related questions (i) and (ii) concerning the asymptotic behavior of the maximum tree extension count $M_n$. In particular, we establish that concentration around

$^1$Using this convention we end up counting each copy of $H$ exactly aut($H$) times, where aut($H$) is the number of automorphisms of $H$ which fix the root.

$^2$We henceforth denote by $(n)_k$ the falling factorial $(n)_k = n(n-1) \cdots (n-k+1)$, as usual.

$^3$A rooted tree $T$ is balanced but not strictly balanced, because $e_F / (v_F - 1) = 1$ for all connected subgraphs $F \subseteq T$ (since these are again trees).
the mean $M_n/\mu_T \xrightarrow{p} 1$ holds when $p(1-p)n \gg \log n$, in which case we also obtain the correct asymptotic expression for $M_n - \mu_T$ (see Theorem 1) and show that the maximum degree essentially determines the behavior of $M_n$ (see Proposition 2). Furthermore, we demonstrate that the aforementioned range of edge probabilities $p = p(n)$ is best possible, since $M_n/\mu_T \xrightarrow{p} 1$ fails when $pn = \Theta(\log n)$ holds (see Corollary 3).

Turning to the details, set $q := 1 - p$ for brevity. As discussed, in the ‘dense’ case $pqn \gg \log n$ the maximum degree $\Delta = \Delta(G_{n,p})$ is known to satisfy the asymptotic behavior (2); see also Proposition 9. Since the degree variance is asymptotically $pqn$, in concrete words the limit (2) says that ‘the maximum degree typically deviates from the mean degree by $\sqrt{2 \log n}$ standard deviations’. Our first main result shows that this strong concentration assertion in fact holds for all maximum rooted tree extension counts.

**Theorem 1** (Maximum for general trees, dense case). Fix a rooted tree $T$, with root degree $a$. If $pqn \gg \log n$, then

$$\frac{M_n - \mu_T}{\sigma_T \sqrt{2 \log n}} \xrightarrow{p} 1, \tag{4}$$

where the variance $\sigma^2_T := \text{Var}_v X_v$ satisfies $\sigma^2_T \sim a^2 \mu^2 q/(pn)$.

En route to Theorem 1 we establish Proposition 2 below, which is stronger in two ways: (i) it allows us to understand the number of $T$-extensions of $v$ for every vertex $v$ as a simple function of the degree of $v$, and (ii) it gives a smaller error. This in particular demonstrates that vertices of near maximum degree determine the asymptotic behavior (4) of $M_n$. We remark that in the regime $pqn \geq C_0 \log n$ the technical condition $C_0 > 1$ implies that the minimum degree is of order $pn$, which somewhat simplifies our arguments.

**Proposition 2.** Fix a rooted tree $T$, with root degree $a$. For any constant $C_0 > 1$ there exists $C > 0$ such that the following holds. If $pqn \geq C_0 \log n$, then with high probability

$$\max_{v \in [n]} |X_{T,v} - d(v)^a (pq)^{\tau-a}| \leq C(pn)^{\tau-1} \sqrt{\log n}, \tag{5}$$

where $d(v)$ denotes the degree of vertex $v$ in the random graph $\mathbb{G}_{n,p}$.

The following simple corollary of Proposition 2 (proved in Section 3) shows that the maximum $M_n$ is no longer concentrated around $\mu_T$ when $pn = \Theta(\log n)$.

**Corollary 3.** Fix a rooted tree $T$. There is a strictly decreasing function $f : (1, \infty) \to (1, \infty)$ such that the following holds. If $pn \sim C \log n$ for some constant $C > 1$, then

$$\frac{M_n}{\mu_T} \xrightarrow{p} f(C) > 1. \tag{6}$$

1.2. Can $M_n$ behave differently for smaller edge probabilities? Question (iii) for maximum tree extension counts asks us to identify a sequence $\alpha_n$ such that $M_n/\alpha_n \xrightarrow{p} 1$, and in view of the results from Section 1.1 we henceforth restrict our attention to the sparser regime $pn \ll \log n$. Here we expect that $\alpha_n$ will be significantly larger than the mean $\mu_T$, but it seems difficult to determine $\alpha_n$ for general trees $T$. The results in this section demonstrate that indeed new complexities emerge in the answer to question (iii): for an interesting class of trees we establish that the form of $\alpha_n$ changes for different ranges of $p = p(n)$, and show that the behavior $M_n$ is not always associated with the maximum degree, i.e., our large deviation type optimization proofs reveal that there are several different mechanisms that can make $M_n$ large (see Theorems 4, 5, and Corollary 6).

To illustrate different types of behaviors that the maximum extension count $M_n$ can have when $pn \ll \log n$, we shall restrict our attention to spherically symmetric trees $T_{a,b}$ of height two, where the root has $a$ neighbors (children), and each of these vertices has $b$
children. To motivate the statement of Theorem 4 below for $T = T_{a,b}$, it is instructive to consider different possible ‘strategies’ for finding a vertex $v$ for which the extension count $X_v = X_{T,v}$ is particularly large. One natural strategy would be to select a vertex $v$ of maximum degree $\Delta$, which for $1 \ll pn \ll \log n$ satisfies (see Proposition 10):

$$\frac{\Delta}{D} \xrightarrow{p} 1 \quad \text{with} \quad D = D(n,p) := \frac{\log n}{\log \log n}.$$  \hspace{1cm} (7)

It is plausible that most neighbors of such a vertex $v$ should have degree $(1 + o(1))pn$, so this strategy ought to produce a vertex $v$ such that, whp (with high probability, i.e., with probability tending to 1 as $n \to \infty$),

$$X_v \geq (1 + o(1))D^a((pn)^{ab}).$$  \hspace{1cm} (8)

Another possible strategy would be to select a vertex $v$ which is adjacent to one or more vertices of large degree, i.e., of order $D$. Since for any $x \in (0,1)$ the ‘probability cost’ associated with having degree around $xD$ turns out to be $n^{-x+o(1)}$ (see Lemma 8), it is usually possible to find a vertex $v$ which has, amongst its neighbors, a set of $a$ neighbors which all have degrees of the form $(1/a + o(1))D$. This strategy thus ought to produce a vertex $v$ such that, whp,

$$X_v \geq (1 + o(1))f_{a,b}(x) D^{ab}. \quad \text{(11)}$$

In fact, the latter strategy is more versatile, as one may consider other sequences which sum to one in place of the sequence $(1/a, \ldots, 1/a)$. Intuitively, $$\Lambda := \bigcup_{k \geq 1} \left\{ (x_1, \ldots, x_k) \in [0, \infty)^k : \sum_{1 \leq i \leq k} x_i \leq 1 \right\}$$

represents the space of possible implementations of this strategy. To evaluate the number of extensions each achieves, we define the function $f_{a,b} : \Lambda \to \mathbb{R}$ by setting

$$f_{a,b}(x_1, \ldots, x_k) := \sum_{\text{distinct } i_1, \ldots, i_a \in [k]} \prod_{j \in [a]} x_{i_j}^b. \quad \text{(10)}$$

Note that $f_{a,b}$ is trivially zero whenever $k < a$. Putting things together, for any vector $x \in \Lambda$ the discussed strategy ought to produce a vertex $v$ such that, whp,

$$X_v \geq (1 + o(1))f_{a,b}(x) D^{ab}. \quad \text{(11)}$$

Naturally, to get the best lower bound one should then optimize $f_{a,b}$ over $\Lambda$.

Our second main result determines the asymptotic behavior of $M_n$ for spherically symmetric trees $T = T_{a,b}$ with $b \geq 2$: it shows that one of the two strategies discussed above is optimal over practically the whole sparse range $1 \ll pn \ll \log n$.

**Theorem 4** (Maximum for spherically symmetric trees $T_{a,b}$, simplified). Fix $T = T_{a,b}$, with $a \geq 1$ and $b \geq 2$. If $(\log n / \log \log n)^{1-1/b} \ll pn \ll \log n$, then

$$\frac{M_n}{D^a((pn)^{ab})} \xrightarrow{p} 1. \quad \text{(12)}$$

If $1 \ll pn \ll (\log n / \log \log n)^{1-1/b}$, then

$$\frac{M_n}{D^{ab}} \xrightarrow{p} \sup_{x \in \Lambda} f_{a,b}(x). \quad \text{(13)}$$

**Remark 1.** Theorem 18 in Section 6 shows that the correct asymptotics in the ‘missing range’ $pn \asymp (\log n / \log \log n)^{1-1/b}$ are obtained by a combination of the above strategies.
The most important case not covered by Theorem 4 corresponds to the two-edge path \( T = T_{1,1} = P_2 \) rooted at an endvertex. In this case Theorem 5 below shows that the first strategy ‘wins’ whenever \( pn \gg \log \log n \), yielding \( M_n/Dpn \xrightarrow{p} 1 \). Here the second strategy only gives \( \Theta(D) \) extensions (since \( a = b = 1 \)), meaning that it is always inferior to the first strategy. Note that in the two strategies discussed above, the vertex \( v \) which maximizes the number of extensions is either itself of (near) maximum degree \( D \) or has neighbors of degree \( \Theta(D) \). Interestingly, in the very sparse regime \( pn \ll \log \log n \) a new optimal strategy emerges for paths \( P_2 \) of length two: the proof of Theorem 5 below shows that the maximum number of extensions can be attained by a vertex of degree \( o(D) \), or, more precisely, by a vertex \( v \) whose degree is asymptotically equal to

\[
\frac{1}{\log \frac{\log \log n}{pn}} \cdot D \approx \frac{\log n}{(\log \log n) \log \frac{\log \log n}{pn}},
\]

and whose neighbors have, on average, degree about \( \log \log n \).

**Theorem 5** (Maximum for paths of length two, sparse case). Let \( T = T_{1,1} \) be a path of length two rooted at one endvertex. If \( 1 \ll pn \ll \log n \), then

\[
\frac{M_n}{\alpha_n} \xrightarrow{p} 1,
\]

where the sequence \( \alpha_n \) satisfies

\[
\alpha_n \approx \begin{cases} 
Dpn & \text{if } \log \log n \ll pn \ll \log n, \\
\log n \log \frac{\log \log n}{pn} & \text{if } 1 \ll pn = O(\log \log n).
\end{cases}
\]

**Remark 2.** Theorem 15 in Section 5 extends Theorem 5 to paths \( P_m \) of fixed length \( m \geq 1 \), in which case the optimal strategy changes more often; see the discussion below (58).

Finally, the following corollary of Theorem 5 (proved in Section 5) covers the remaining spherically symmetric trees \( T_{a,b} \) with \( b = 1 \) that were excluded so far.

**Corollary 6.** Fix \( T = T_{a,1} \), with \( a \geq 1 \). If \( 1 \ll pn \ll \log n \), then

\[
\frac{M_n}{\alpha^2_n} \xrightarrow{p} 1,
\]

where the sequence \( \alpha_n \) is as in Theorem 5.

### 1.3. Background and discussion.

The study of subgraph counts in random graphs has become an extremely well-established area of research \[Bol81, Ruc88, JOR04, JW16, SW19, HMS22\]. Extension counts are an important and natural variant, which frequently arises in many probabilistic proofs and applications, including zero-one laws in random graphs \[SS88, LS91\], games on random graphs \[LP10\], random graph processes \[BK10, BW19\], and random analogues of classical extremal and Ramsey results \[SS18, BK19\].

For strictly balanced rooted graphs \( H \), the concentration of the maximum extension count \( M_n = M_{H,n} \) around the mean count \( \mu_H \) is fairly well understood.\(^4\) Indeed, for such graphs Spencer \[Spe90\] proved that \( M_n/\mu_H \xrightarrow{p} 1 \) when \( \mu_H \gg \log n \), and asked whether this condition on \( \mu_H \) was necessary. Šileikis and Warnke \[ŠW22\] answered this question, by showing that \( M_n/\mu_H \xrightarrow{p} 1 \) breaks down when \( \mu_H = \Theta(\log n) \).

For rooted graphs \( H \) that are not strictly balanced our understanding remains unsatisfactory, and we are not aware of a general formula for the order of magnitude of \( M_n \) for all edge probabilities \( p = p(n) \) of interest. In particular, as pointed out in \[ŠW22\],

\[^4\]We remark that \[Spe90\] and \[ŠW22\] both treat more general extension counts, where the root may consist of a set of vertices (while we focus on the case where the root consists of a single vertex).
the behavior of the maximum extension count $M_n$ can be quite different if $H$ is not strictly balanced. For example, for graphs that are not balanced it can even happen\textsuperscript{5} that $M_n = n^{o(1)}$ despite $\mu_H \ll 1$. In view of this it remains an interesting open problem to find, for all root graphs, a suitable sequence $\alpha_n$ such that $M_n/\alpha_n$ is tight.

In Sections 1.1–1.2 we have seen that in some cases it is possible to find a sequence $\alpha_n$ such that $M_n/\alpha_n \to 1$, or, even better, to find yet another sequence $\beta_n$ such that

$$\frac{M_n - \alpha_n}{\beta_n} \xrightarrow{p} 1.$$  \hfill (17)

It remains an intriguing open problem to understand for which rooted graphs and for which edge probabilities such detailed results hold. To stimulate more research into this circle of problems, we propose the following modest generalization of Theorem 1.

**Problem 1.** Determine for which connected rooted graphs $H$ the two natural conditions

$$\Phi_H := \min_{G \subseteq H} \mu_G \gg \log n$$

and $p \ll 1$ together imply that

$$\frac{M_n - \mu_H}{\sigma_H \sqrt{2 \log n}} \xrightarrow{p} 1.$$  \hfill (18)

To motivate why (18) is plausible for many rooted graphs, first note that the standard variance estimate $\sigma_H^2 \asymp \Phi_H^2$ (see [SW22, eq.(10)]) and the assumption $\Phi_H \gg \log n$ together imply that $\sigma_H \sqrt{2 \log n} \ll \mu_H$, so we are in the moderate deviations regime. Based on the fact that $X_v$ is asymptotically normal (see [SW22, Claim 17(ii)]), the hope is then (i) that the upper tail satisfies $-\log P(X_v \geq \mu_H + x) \sim x^2/(2\sigma_H^2)$ and (ii) that the union bound over the $n$ choices of $v$ is not wasteful, which together lead to (18). In concrete words, Theorem 1 shows that this heuristic is correct for rooted trees, and we believe that it is also correct for many other rooted graphs. However the general case is somewhat delicate, since this heuristic can fail: see [SW22, Proposition 2] for a counterexample. Of course, it would also be interesting to generalize Problem 1 to graphs with $r$ root vertices, defining $M_n$ as the maximum of extension counts over all $r$-tuples (here the natural denominator in (18) would then be $\sigma_H \sqrt{2r \log n}$).

Occasionally it is possible to further refine (17) by finding a non-degenerate limit distribution of the random variable $(M_n - \alpha_n - \beta_n)/\gamma_n$ for certain sequences $\alpha_n$ and $\beta_n$ satisfying (17), and a further sequence $\gamma_n$. For the maximum degree, such results were obtained by Ivchenko [Ivc73] and Bollobás [Bol80] in the 1970s and 1980s; see also [RZ23]. For rooted cliques, such results were also recently obtained by Isaev, Rodionov, Zhang and Zhukovskii [IRZZ], for some restricted range of edge probabilities $p = p(n)$. So far, in all known cases the limiting distribution is the standard Gumbel distribution, and it would be of interest to establish this for a larger family of rooted graphs.

1.4. Organization of the paper. In Section 2 we present some preliminary random graph theory results. In Section 3 we prove Theorem 1 for arbitrary trees $T$ in the ‘dense’ case $pqn \gg \log n$. In Section 4 we prove two lemmas which allow us to consider extension counts in random trees rather than random graphs, and these auxiliary results are subsequently used to prove our main results in the ‘sparse’ case $1 \ll pn \ll \log n$: in Section 5 we prove Theorem 5 for paths $P_m$ of any fixed length $m \geq 1$ (see Theorem 15), and in Section 6 we give the more involved proof of Theorem 4 for spherically symmetric trees $T_{a,b}$ of height two in the entire range $1 \ll pn \ll \log n$ (see Theorem 18). Finally, in Section 7 we demonstrate that our methods also carry over to minimum rooted tree extension counts, i.e., to the random variable $\min_{v \in [n]} X_v$ (see Theorem 29).

\textsuperscript{5}Consider for example a rooted triangle with a pendant edge added at one of the non-root vertices, with $p = n^{-\gamma}$ for some $\gamma \in (3/4, 1)$. By rooting at a vertex of which is in a triangle of $G_{n,p}$, it then is straightforward to see that w.h.p $M_n \geq \Theta(pn) = \Theta(n^{1-\gamma})$, despite $\mu_H = \Theta(n^3p^3) \ll 1$. 

2. Preliminaries

In this preparatory section we introduce basic concepts and results on the binomial distribution and the degrees in $G_{n,p}$. We also provide an asymptotic formula for the variance of the number of $T$-extensions. Since all proofs are more or less routine (and some of them even follow from textbook results [Bol01]), we defer them to Appendix A to avoid clutter. On a first reading the reader may wish to skip straight to Section 3.

2.1. Asymptotics of binomial probabilities. We recall rather universal upper and lower bounds for binomial probabilities. Let $q := 1 - p$ and $\xi \sim \text{Bin}(n,p)$. To this end we recall the large deviation rate function for the Poisson distribution, defined as

$$\phi(x) := (1 + x) \log(1 + x) - x, \quad x \in [-1, \infty),$$

with $\phi(-1) = 1$ defined as the right limit (as usual). We will use the following well-known Chernoff bound (see [JLR00, Theorem 2.1]) for the upper tail:

$$\mathbb{P}(\xi \geq (1 + \eta)pn) \leq \exp\left(-pm\phi(\eta)\right), \quad \eta \geq 0. \tag{20}$$

Noting that $\phi(\eta) \geq \phi(\eta) - 1 = (1 + \eta) \log((1 + \eta)/e)$, inequality (20) yields the following simple upper bound, which is often useful when $x > epn$:

$$\mathbb{P}(\xi \geq x) \leq \exp\left(-x \log \frac{x}{epn}\right), \quad x > 0. \tag{21}$$

The following proposition gives a lower bound to the point probability. When (22) is used as a lower bound for the upper tail, typically the $\log k$ term is negligible, in which case it matches the upper bound (20).

Proposition 7. Let $\xi \sim \text{Bin}(n,p)$. Let $k = (1 + \eta)pn$ be an integer. If $1 \ll k \ll \sqrt{n}$ and $pn \ll \sqrt{n}$, then

$$\mathbb{P}(\xi = k) \geq \exp\left(-pm\phi(\eta) + O(\log k)\right). \tag{22}$$

It is also convenient to have the following inequalities (which follow from inequalities above) when one aims at probabilities of the form $n^{-c}$.

Lemma 8. Let $D = D(n,p)$ be as defined in (7). Assume that $pn \ll \log n$. Assume that $\alpha = \alpha(n) > 0$ satisfies $|\log \alpha| = o\left(\log \frac{\log n}{pn}\right)$, which includes any constant $\alpha > 0$. Then the random variable $\xi \sim \text{Bin}(n,p)$ satisfies

$$\mathbb{P}(\xi \geq \alpha D) \leq n^{-o(1+o(1))}. \tag{23}$$

Furthermore, under the additional assumption $pn \geq 1$, for any constants $\alpha, \varepsilon > 0$ we have

$$\mathbb{P}(\alpha D \leq \xi < (\alpha + \varepsilon)D) \geq n^{-\alpha + o(1)}. \tag{24}$$

2.2. Extremal degrees of binomial random graphs. We now recall some well-known results about the maximum and minimum degrees in the binomial random graph $G_{n,p}$, denoted by $\Delta = \Delta(G_{n,p})$ and $\delta = \delta(G_{n,p})$ respectively. Proposition 9 gives detailed information about the extremal degrees degrees in the ‘dense’ case $pqn \gg \log n$.

Proposition 9. If $pqn \gg \log n$, then

$$\frac{\Delta - pn}{\sqrt{2pqn \log n}} \xrightarrow{p} 1 \quad \text{and} \quad \frac{pn - \delta}{\sqrt{2pqn \log n}} \xrightarrow{p} 1. \tag{25}$$

In the ‘sparse’ case $1 \ll pn = O(\log n)$, Proposition 10 states that the asymptotics of the maximum degree $\Delta$ of $G_{n,p}$ is more involved (and, when $pn \ll \log n$, then the minimum degree of $G_{n,p}$ is in fact typically zero, see Section 7). Asymptotic properties of the inverse $\phi^{-1}$ of the function $\phi$ defined in (19) are discussed in Remark 3 below.
Proposition 10. If $1 \ll pn = O(\log n)$, then
\[
\frac{\Delta - pn}{\alpha_n} \xrightarrow{p} 1,
\] (26)
where $\alpha_n := pn\phi^{-1}\left(\frac{\log n}{pn}\right)$. If furthermore $pn \ll \log n$, then
\[
\alpha_n \sim \frac{\log n}{\log \frac{\log n}{pn}} \gg pn.
\] (27)

Remark 3. The function $\phi : [-1, \infty) \to [0, \infty)$ satisfies the following well-known asymptotics that can be proved by basic calculus:
\[
\phi(x) \sim \begin{cases} x^2/2 & \text{if } x \to 0, \\ x \log x & \text{if } x \to \infty. \end{cases}
\] (28)

When restricted to $[0, \infty)$, $\phi$ is an increasing bijection onto $[0, \infty)$ and thus has an increasing inverse $\phi^{-1} : [0, \infty) \to [0, \infty)$. In view of (28), the function $\phi^{-1}$ satisfies
\[
\phi^{-1}(y) \sim \begin{cases} \sqrt{2y} & \text{if } y \to 0, \\ \frac{y}{\log y} & \text{if } y \to \infty. \end{cases}
\] (29)

2.3. Expectation and variance of tree extension counts. Recall that $X_v = X_{T,v}$ denotes the number of $T$-extensions of $v$. For trees $T$ we now record the (routine) asymptotics of the expectation and variance of $X_v$.

Proposition 11. Fix a rooted tree $T$, with root degree $a$. Let $\mu_T$ and $\sigma_T^2$ denote the expectation and variance of $X_v = X_{T,v}$, for some vertex $v \in [n]$. If $pn \gg 1$, then
\[
\mu_T = (pn)^{\varepsilon_T} (1 + O(1/n)) \quad \text{and} \quad \sigma_T^2 \sim a^2 \mu_T^2 q/(pn).
\] (30)

3. TREES: THE DENSE CASE

In this section we deal with extension counts of an arbitrary rooted tree $T$ in the ‘dense’ case $pqn \gg \log n$. In particular, in Section 3.1 we deduce Theorem 1 and Corollary 3 from the key result Proposition 2, which is subsequently proved in Section 3.2.

3.1. Proof of the main result for trees: Theorem 1 and Corollary 3. In the following proofs of Theorem 1 and Corollary 3, to avoid clutter we shall write $o_p(a_n)$ for a sequence of random variables $X_n$ such that $X_n/a_n \xrightarrow{p} 0$, as usual (see [JLR00, p. 11]).

Proof of Theorem 1. The claimed variance asymptotics of $\sigma_T^2$ hold by Proposition 11. Gearing up towards (4), set $\gamma := \sqrt{2\gamma \log n/(pn)}$. Proposition 9 implies that
\[
\max_{v \in [n]} d(v) = pn \left(1 + \gamma \left(1 + o_p(1)\right)\right).
\]
Considering the error term in (5), note that condition $pqn \gg \log n$ implies
\[
C(pn)^{\varepsilon_T - 1} \sqrt{\log n} \ll \gamma(pn)^{\varepsilon_T}
\]
and $\gamma \to 0$. Using the conclusion (5) of Proposition 2 it then follows that
\[
M_n = \max_{v \in [n]} X_v = (pn)^{\varepsilon_T} \cdot \left(1 + \gamma((1 + o_p(1)) \right)^a + o_p(\gamma(pn)^{\varepsilon_T})
\]
\[= (pn)^{\varepsilon_T} (1 + a\gamma + o_p(\gamma)).
\]
This implies (4) by noting (see (30)) that $\mu_T = (pn)^{\varepsilon_T} (1 + O(1/n)) = (pn)^{\varepsilon_T} (1 + o(\gamma))$ and
\[a\gamma(pn)^{\varepsilon_T} \sim \sqrt{\frac{a^2 \gamma^2 \log n}{pn}} \mu_T \sim \sigma_T \sqrt{2 \log n},
\]
completing the proof of Theorem 1.

Proof of Corollary 3. With foresight, we define $f(C) := (1 + \phi^{-1}(1/C))^a$, where $\phi^{-1}$ is the inverse of the function $\phi$. The desired properties of $f$ follow from Remark 3. Proposition 10 then implies (using continuity of $\phi^{-1}$) that
\[
\max_{v \in [n]} d(v) = pn \cdot (1 + \phi^{-1}(1/C) + o_p(1)),
\]
whereas the error term in (5) is $o((pn)^{εr})$. Using the conclusion (5) of Proposition 2, in view of $f(C) \geq 1$ it then follows that
\[
M_n = \max_{v \in [n]} X_v = (pn)^{εr} \cdot (1 + \phi^{-1}(1/C) + o_p(1))^a + o_p((pn)^{εr})
\]
\[
= (pn)^{εr} \cdot f(C) \cdot (1 + o_p(1)),
\]
which together with $(pn)^{εr} \sim \mu_T$ establishes (6), completing the proof of Corollary 3. □

3.2. Proof of main technical result for trees: Proposition 2. The following proof of Proposition 2 is based on induction. Here the base case is the core of the matter: its proof hinges on the following variant of the bounded differences inequality due to Warnke [War16], which conveniently takes into account (a) that the typical one-step changes can be much smaller than the worst case ones and (b) that the underlying independent random variables are binary. (To clarify: Lemma 12 follows from [War16, Theorem 1.3 and eq. (1.5)], by setting $c_k = r, d_k = R, γ_k = r/R$ and $p_k = p$.)

Lemma 12. Let $X = (ξ_1, \ldots, ξ_N) \in \{0, 1\}^N$ be a family of independent random variables with $P(ξ_k = 1) = p$ for all $1 \leq k \leq N$. Let $f : \{0, 1\}^N \rightarrow \mathbb{R}$ be a function. Let $Γ \subseteq \{0, 1\}^N$ be an event. Assume that there are $R \geq r \geq 0$ such that
\[
|f(x) - f(\bar{x})| \leq \begin{cases} r & \text{if } x \in Γ, \\ R & \text{otherwise}, \end{cases}
\]
whenever $x, \bar{x} \in \{0, 1\}^N$ differ in exactly one coordinate. Then, for all $t \geq 0$,
\[
P(|f(X) - E f(X)| \geq t) \leq 2 \cdot \exp \left( - \frac{t^2}{2Npr^2 + 4rt} \right) + P(X \notin Γ) \cdot 2NR/r.
\]

Proof of Proposition 2. We start with the fact that our assumption $pn \geq C_0 \log n$, with constant $C_0 > 1$, implies (see [Bol01, Exercise 3.4]) that there are two constants $0 < C_1 < 1 < C_2$ depending on $C$ such that
\[
P(d(v) \in [s_1, s_2] \text{ for all } v \in [n]) = 1 - o(1) \quad \text{with } s_j := C_j pn.
\]

Let $Z_{T,v}$ denote the number of graph homomorphisms of $T$ into $G_{n,p}$ mapping the root of $T$ to $v$. In our counting arguments it will be convenient to consider $Z_{T,v}$ instead of $X_{T,v}$, since we then do not need to worry about whether certain vertices coincide. We trivially have $Z_{T,v} \geq X_{T,v}$ (since $Z_{T,v}$ includes non-injective homomorphisms), and we claim that typically $Z_{T,v} \approx X_{T,v}$ holds, more precisely that
\[
\max_{v \in [n]} |Z_{T,v} - X_{T,v}| \leq (pn)^{εr-1}(\log n)^{1/9} \quad \text{whp.}
\]
Furthermore, writing $ε := \sqrt{\log n/pn}$, we claim that, for some constant $C_T > 0$,
\[
\max_{v \in [n]} \left| \frac{Z_{T,v}}{(pn)^{εr-a}} - 1 \right| \leq C_T ε \quad \text{whp},
\]
which readily implies the desired estimate (5) with $C = C_T^a C_T + 1$, by applying the triangle inequality and the degree bound $d(v) \leq s_2 = C_2 pn$ from (33).
Throughout the remainder of the proof we will define further positive constants $C_3, C_4,$ and $C_5,$ where each $C_i$ may depend on $C_1, \ldots, C_{i-1}$ and $T.$

For the proof of inequality (34), we shall bound non-injective homomorphisms in terms of the maximum degree $\Delta = \max_{v \in [n]} d(v),$ exploiting the loss of freedom due to repeated vertices. In particular, every homomorphism counted by $Z_{T,v} - X_{T,v} \geq 0$ maps the $v_T - 1 = e_T$ non-root vertices to $k \leq e_T - 1$ vertices, and so the image of this mapping contains a $U$-extension of $v,$ where $U$ is some rooted tree with $u_V = k + 1$ vertices and $e_U = k$ edges. Taking all such rooted trees $U$ into account, by iteratively choosing the edges of $U$ in $\mathbb{G}_{n,p}$ (each adjacent to the root vertex $v$ or an already chosen edge) we infer via the maximum degree bound $\Delta \leq s_2 = C_2 pn$ from (33) that, whp,

$$\max_{v \in [n]} |Z_{T,v} - X_{T,v}| \leq \sum_{0 \leq k \leq e_T - 1} O(\Delta^k) \leq O((pn)^{e_T - 1}) \ll (pn)^{e_T - 1} (\log n)^{1/9},$$

establishing the desired bound (34), as claimed.

We proceed to the proof of inequality (35), for which we will use induction on the height $h$ of the tree $T,$ the base case being $h \in \{0, 1, 2\}.$ The case $h = 0$ is trivial and for $h = 1$ the formula $Z_{T,v} = d(v)^a = d(v)^a (pn)^b$ holds deterministically. Deferring the proof of the remaining base case $h = 2$ (which is most of the work), we now turn to the induction step for height $h \geq 3,$ where we assume that (35) holds for all trees $T$ of height at most $h - 1.$ Now, for a tree $T$ of height $h \geq 3,$ let $u_1, \ldots, u_a$ be the children of the root of $T,$ and, for $i \in [a],$ let $T_i$ denote the rooted tree consisting of the root $u_i$ and all its descendants in $T.$ For notational convenience, set

$$a_i \equiv \deg_{T_i}(u_i) \quad \text{and} \quad b \equiv \sum_{i \in [a]} a_i.$$ 

Denoting by $N(v)$ the set of neighbors of $v$ in $\mathbb{G}_{n,p},$ using induction we infer that, whp,

$$Z_{T,v} = \sum_{v_1, \ldots, v_a \in N(v)} \prod_{i \in [a]} Z_{T_i,v_i}$$

$$= \sum_{v_1, \ldots, v_a \in N(v)} \prod_{i \in [a]} d(u_i)^{a_i} (pn)^{a_i - a} (1 \pm C_{T_i} \varepsilon)$$

$$= \left( \sum_{v_1, \ldots, v_a \in N(v)} \prod_{i \in [a]} d(u_i)^{a_i} \right) \cdot (pn)^{a - b} \prod_{i \in [a]} (1 \pm C_{T_i} \varepsilon).$$

(36)

for all $v \in [n].$ Denote by $T'$ the subtree of $T$ induced by vertices of depth at most two, so that $e_{T'} = a + b.$ Observe that the first factor in (36) is precisely $Z_{T',v}$ and using the induction hypothesis again, this time for height 2, it follows that, whp,

$$Z_{T,v} = (1 \pm C_{T'} \varepsilon) d(v)^a (pn)^b \cdot (pn)^{e_T - a - b} \prod_{i \in [a]} (1 \pm C_{T_i} \varepsilon)$$

for all $v \in [n],$ which implies $Z_{T,v} = (1 \pm C_{T'} \varepsilon) d(v)^a (pn)^{e_T - a}$ with a suitable constant $C_T > 0$ for large enough $n$ (since $\varepsilon \to 0$), proving the induction step.

We now return to the base case of height $h = 2,$ which is the core of the matter. Here we find it convenient to focus on injective homomorphisms counts $X_{T,v}$ (and then use (34) to transfer to $Z_{T,v}$). Fix a tree $T$ of height two. Given an integer $s,$ we define

$$\mu_s \equiv \mathbb{E}[X_{T,v} \mid d(v) = s] = (s)_a \cdot p^{e_T - a} (n - a - 1)^{e_T - a}.$$ 

(37)

A suitable constant $C_4 > 0$ will be determined later, while at the moment we set

$$C_5 \equiv 8 \sqrt{C_2 (1/C_1)^{e_T}} C_4,$$ 

(38)
and then define the ‘good’ event
\[ G_v := \{ |X_{T,v} - \mu_d(v)| < C_5 \varepsilon s_1^{eT} \}. \] (39)

To complete the proof, we claim that it suffices to show the following inequalities:
\[ \max_{v \in [n]} \mathbb{P}(-G_v, d(v) \notin [s_1, s_2]) \ll \frac{1}{n}. \] (40)

Indeed, to see how inequalities (33), (40) and (41) imply the base case, note that a standard union bound argument combined with (33) and (41) readily gives
\[ \mathbb{P} \left( \bigcup_{v \in [n]} \{ \neg G_v \text{ or } d(v) \notin [s_1, s_2] \} \right) \leq o(1) + \sum_{v \in [n]} \mathbb{P}(-G_v, d(v) \in [s_1, s_2]) \ll 1. \]

Hence whp the events \( G_v \) and \( d(v) \in [s_1, s_2] \) hold simultaneously for all \( v \in [n] \). Combining this with inequalities (34) and (40), using \( s_1 \leq pn \) and \( (pn)^{eT-1}(\log n)^{1/9} \ll \varepsilon (pn)^{eT} \) it follows that, whp, for all \( v \in [n] \) we have
\[ Z_{T,v} = \mu_d(v) + O_5 s_1^{eT} \pm (pn)^{eT-1}(\log n)^{1/9} = d(v)^a (pn)^{eT-a} (1 \pm 2C_5 \varepsilon), \]

establishing the desired bound (35) with \( C_T := 2C_5 \), as claimed.

In the remainder we prove the claimed inequalities (40) and (41). We start with inequality (40), where by (37) we have \( \mu_s = (s)_a \cdot p^{n-a} (n-a-1)_{eT-a} \). For all \( s \geq s_1 = C_1 pn \) we uniformly have \( (s)_a = s^a (1 - O(1/pn)) \), which together with \( (n-a-1)_{eT-a} = n^{eT-a} (1 - O(1/n)) \) and \( \varepsilon \gg 1/pn \) readily establishes the claimed inequality (40).

Finally, we turn to the proof of inequality (41), where by symmetry it suffices to consider the vertex \( v = 1 \). Fix \( s \in [s_1, s_2] \). By symmetry \( \mathbb{P}(\neg G_1 | d(1) = s) = \mathbb{P}(\neg G_1 | N(1) = S) \), where \( S := \{2, \ldots, s + 1\} \). Henceforth we consider the conditional probability space with respect to the event \( N(1) = S \), and use the shorthand \( \mathbb{P}_s, \mathbb{E}_s \) for the associated probabilities and expectations. In particular, we have \( \mu_s = \mathbb{E}_s X_{T,1} \) and
\[ \mathbb{P}(\neg G_1 | d(1) = s) = \mathbb{P}_s(|X_{T,1} - \mathbb{E}_s X_{T,1}| \geq C_5 \varepsilon s_1^{eT}). \] (42)

Note that this conditional probability space corresponds to \( \binom{n-1}{2} \) independent binary random variables, each with success probability \( p \) (which encode the status of the pairs \( xy \) with \( x, y \neq v \), i.e., whether they are an edge or not). In fact, \( X_{T,1} \) is already determined by
\[ N := s(n-1-s) + \binom{s}{2} \leq sn \leq s^2 n \leq C_3 pm^2 \] (43)
of these random variables, namely the status of the pairs \( xy \) with at least one element in \( S \); we denote these independent random variables by \( X = (\xi_1, \ldots, \xi_N) \in \{0, 1\}^N \). Hence there is a (deterministic) function \( f : \{0, 1\}^N \rightarrow \mathbb{R} \) such that \( X_{T,1} = f(X) \).

In the following we prepare for applying Lemma 12 to \( X_{T,1} = f(X) \). For the Lipschitz condition (31), we need to control by how much \( X_{T,1} \) changes if we alter the status of one pair of vertices, i.e., whether it is an edge or not. To this end, given a graph \( G \) with \( N(1) = S \), let \( X_{T,1,G,uv} \) denote the number of \( T \)-extensions of vertex 1 in \( G \) containing the edge \( uv \). Note that any such \( T \)-extension must map some child \( w \) of the root to a vertex \( z \in \{u, v\} \) and then some child of \( w \) to \( \{u, v\} \setminus \{z\} \). To bound \( X_{T,1,G,uv} \) from above, we can thus first choose which vertices we map to \( u \) and \( v \) (in at most \( 2 \sum_{i \in [a]} a_i \leq 2eT \) ways), and then iteratively map the remaining \( vT - 3 = eT - 2 \) vertices of \( T \) in a suitable order (to a vertex adjacent to the root vertex 1 or a vertex already mapped to), yielding
\[ X_{T,1,G,uv} \leq 2eT \cdot \left[ \max_{w \in S \cup \{1\}} d_G(w) \right]^{eT-2}, \] (44)
where \( d_G(w) \) denotes the degree of \( w \) in \( G \). As the degree \( d(1) = s \leq s_2 = C_2pn \) of vertex 1 is fixed (in the conditional probability space), we define the ‘good’ event

\[
\Gamma := \left\{ \max_{v \in S} d(v) \leq C_3pn \right\} \quad \text{with} \quad C_3 := e^{2eT} + 1.
\]

Note that \( \Gamma \) is determined by \( X = (\xi_1, \ldots, \xi_N) \), and that vertex \( w \in S \) has degree \( d(w) = 1 + Y_w \), where \( Y_w \sim \text{Bin}(n-2, p) \). Using a standard union bound argument over all \( w \in S \) and the inequality \( pn \geq pqn \geq C_0 \log n \geq \log n \), by applying the Chernoff bound (21) to \( Y_w \) with \( x = e^{2eT}pn \) (since \( d(w) = 1 + Y_w \geq C_3pn \) implies \( Y_w \geq 1 \)) it follows that

\[
\mathbb{P}_s(X \not\in \Gamma) \leq |S| \cdot \left( \frac{ep(n-2)}{e^{2eT}pn} \right)^{e^{2eT}pn} \leq n \cdot e^{-e^{2eT}pn} \ll n^{-5eT}. \quad (45)
\]

Next we claim that, whenever \( x, \tilde{x} \in \{0, 1\}^N \) differ by at most one coordinate, then

\[
|f(x) - f(\tilde{x})| \leq \begin{cases} C_4(pn)^{e^{eT}-2} & \text{if } x \in \Gamma, \\ n^{eT} & \text{otherwise,} \end{cases} \quad (46)
\]

where \( C_4 := 2e^{eT}(\max\{C_3, C_2\})^{e^{eT}-2} \). The second bound \( n^{eT} \) in (46) follows from the coarse bounds \( 0 \leq f(x) \leq n^{eT-1} = n^{eT} \). For the first bound in (46), we imagine that we take \( x \in \Gamma \) and then flip one coordinate to obtain \( \tilde{x} \in \{0, 1\}^N \), which corresponds to removing or adding one edge, say \( uv \). After adding the edge \( uv \) the degrees in the resulting graph \( G \) satisfy, by exploiting the degree property \( x \in \Gamma \) and \( d(1) = s \leq s_2 = C_2pn \),

\[
\max_{w \in S \cup \{1\}} d_G(w) \leq \max\{C_3pn + 1, C_2pn\} \leq 2 \max\{C_3, C_2\}pn,
\]

which together with (44) readily establishes (46).

We are now ready to apply Lemma 12 to \( X_{T,1} = f(X) \) with parameters

\[
r := C_4(pn)^{e^{eT}-2}, \quad R := n^{eT} \quad \text{and} \quad t := C_5 \xi \hat{S}_1^{eT}. \quad (47)
\]

Using \( s_1 \geq C_1pn \) and \( N \leq C_2pn^2 \) (see (43)) together with \( \varepsilon = \sqrt{\log n}/pn \ll 1 \), we now see that we defined \( C_5 > 0 \) in (38) so that the exponent in inequality (32) is at least

\[
\frac{t^2}{8Npn^2 + 4rt} \geq \min \left\{ \frac{t^2}{16Npn^2}, \frac{t}{8r} \right\} \geq \min \left\{ \frac{C_4^{2eT}(C_5 \xi)^2(pm)^2}{16C_2C_4^2}, \frac{C_5 \xi C_4^{eT}(pm)^2}{8C_4} \right\} \geq 2 \log n. \quad (48)
\]

Applying the typical bounded differences inequality (32), by combining the exponent estimate (48) with the error probability estimate (45) it follows that

\[
\mathbb{P}_s(|X_{T,1} - \mathbb{E}_sX_{T,1}| \geq C_5 \xi \hat{S}_1^{eT}) \leq 2 \cdot e^{-2\log n} + o(n^{-5eT}) \cdot O(n^{2+eT}) \ll 1/n,
\]

which together with (42) completes the proof of inequality (41) and thus Proposition 2. \( \square \)

4. Trees: Reduction to Random Trees in the Sparse Case

In this section we establish two auxiliary lemmas that will conceptually simplify the proofs of Theorems 15 and 18 for rooted paths and spherically symmetric trees in Sections 5 and 6: in the ‘sparse’ case \( 1 \ll pn \ll \log n \), Lemmas 13 and 14 below allow us to focus on the tails of extension counts in random trees, which are much easier to handle than extension counts in random graphs where there are more dependencies.

Recall that a Galton–Watson tree \( T_{n,p} \) with offspring distribution \( \xi \sim \text{Bin}(n, p) \) is defined recursively, starting with the root vertex and giving each vertex a random set of children, the number of which is an independent copy of \( \xi \).
For a ‘large’ rooted tree $T$ and a ‘small’ rooted tree $T$, let $f_T(G)$ be the number of $T$-extensions of the root in $T_{n,p}$ (i.e., injective homomorphisms from $T$ to $G$ that map the root of $T$ to the root of $G$). In the next lemma we obtain a lower bound on the number of maximum number of $T$-extensions in $G_{n,p}$ via a lower bound on the tail of the extension count in $T_{n,p}$.

**Lemma 13** (Reduction to random tree: lower bound). Let $T$ be a rooted tree of height $h \geq 1$. Let $p = p(n) \in (0, 1)$ satisfy $1 \ll pn \leq \log n$. There exists a sequence $n^* = n^*(n) \sim n$ such that if a sequence $k = k(n)$ satisfies

$$\Pr (f_T(T_{n^*,p}) \geq k) \gg (\log n)^{h+1}/n,$$

then with high probability some vertex has at least $k$ many $T$-extensions in $G_{n,p}$.

The purpose of the second lemma is to obtain an upper bound on the maximum number of $T$-extensions in $G_{n,p}$ via an upper bound on the tail of the extension count in $T_{n,p}$. The assumption on $k$ will turn out to be negligible in our applications, because in the range $1 \ll pn \ll \log n$ the maximum degree is of order $D \gg pn$ (see (7)) and the maximum count of extensions will thus be of higher order than the mean $\mu_T \sim (pn)^{e_T}$.

**Lemma 14** (Reduction to random tree: upper bound). Fix $c \in (0, 1]$. Let $T$ be a rooted tree of height $h \geq 1$. Let $p = p(n) \in (0, 1)$ satisfy $pn \gg 1$. If a sequence $k = k(n)$ satisfies $k \geq ((1 + c)pn)^{e_T}$ and

$$\Pr (f_T(T_{n,p}) \geq k) \ll 1/n,$$

then with high probability every vertex has at most $k$ many $T$-extensions in $G_{n,p}$.

### 4.1. Proofs of Lemma 13 and 14

The proofs of Lemmas 13 and 14 are both based on coupling arguments, using an exploration process to relate the neighborhood structure in the random graph $G_{n,p}$ to the Galton–Watson tree $T_{n,p}$. More precisely, given a vertex $v \in [n]$, we explore a subtree of $G_{n,p}$ rooted at $v$ as follows. At each step $i \geq 0$, the set $L_i$ will be a subset of the leaves of the partially discovered tree, and $S_i$ will be vertices in the tree that can potentially be connected to some leaf. We start with

$$L_0 := \{v\} \quad \text{and} \quad S_0 := [n] \setminus \{v\}.$$

At each step $i \geq 0$, we pick a vertex $v_i$ in $L_i$ of minimal distance from $v$ and expose the edges between $v_i$ and $S_i$ (their number has distribution $\text{Bin}(|S_i|, p)$). Then we remove $v_i$ from $L_i$, and transfer the new neighbors of $v_i$ from $S_i$ to $L_i$, i.e., set

$$L_{i+1} := (L_i \setminus \{v_i\}) \cup N(v_i, S_i) \quad \text{and} \quad S_{i+1} := S_i \setminus N(v_i, S_i),$$

where $N(u, S)$ denotes the neighbors in $G_{n,p}$ of a vertex $u$ in a set $S$. Repeating this step until $L_i$ has no vertices of depth smaller than $h$, we obtain a tree of height at most $h$ which we call the $h$-neighborhood subtree of $v$ in $G_{n,p}$. Note that the edge set of this tree equals $\cup_i(\{v_i\} \times N(v_i, S_i))$ by construction.

**Proof of Lemma 13.** Let $\Delta$ denote the maximum degree of $G_{n,p}$. By Proposition 10 and monotonicity of function $\phi^{-1}$ there is $D = \Theta(\log n)$ such that whp $\Delta \leq D$. With foresight, set

$$n^* := \lfloor n - n/D\rfloor \quad \text{and} \quad n' := \lceil n/(2D^{h+1})\rceil.$$

We define a sequence of vertex-disjoint rooted subtrees $T_1, T_2, \ldots$ (of random finite length) in $G_{n,p}$ as follows. Pick an arbitrary vertex $u_1 \in [n]$, and let $T_1$ be the $h$-neighborhood tree of $u_1$ in $G_{n,p}$, as defined above. Note that the ‘leftover’ subgraph of $G_{n,p}$ induced by the vertex set $[n] \setminus V(T_1)$, has the same distribution as $G_{n-|V(T_1)|, p}$ (to see this, note that the exploration process only looked at edges with at least one endpoint in $V(T_1)$, so all remaining edges are still present independently with probability $p$). Hence in this
leftover graph we can pick an arbitrary vertex $u_2$, and let $T_2$ be its $h$-neighborhood tree in $G_{n-|V(T_1)|,p}$, as defined above. We repeat the procedure (remove the vertices of the last tree; pick an arbitrary vertex; explore its $h$-neighborhood subtree) until we run out of vertices, obtaining a sequence of disjoint rooted trees $T_1, T_2, \ldots$, as desired.

Let $T_{n^*,p,h}$ be the subtree of $T_{n^*,p}$ consisting of vertices of depth at most $h$. We now claim that we can couple this sequence $T_1, T_2, \ldots$ of trees with independent copies $U_1, \ldots, U_{n'}$ of the tree $T_{n^*,p,h}$, with the property that

$$\Delta \leq D \text{ implies that for every } j \in [n'], \text{ the tree } T_j \text{ exists and satisfies } U_j \subseteq T_j.$$

To see that such a coupling exists, we need to check that as long as (i) we have not completed constructing all trees $T_1, \ldots, T_{n'}$ and (ii) we have not revealed more than $D$ edges incident to the same vertex, we still have enough space to sample another set of $\binom{n^*}{h}$-distributed children. In other words, we need to check that $|S_i| \geq n^*$ until the construction of $T_{n^*,p,h}$ is completed. Note that whenever $\Delta \leq D$, each tree $T_j$ spans at most $\sum_{k=0}^{h} D^k \leq 2D^h$ vertices, and thus the trees $T_1, \ldots, T_{n'}$ exist and together span at most $2n'D^h \leq n/D$ vertices. This means that as long as we have not discovered a vertex with more than $D$ neighbors, we still have $|S_i| \geq n-n/D \geq n^*$; before exposing the edges between $v_i$ and $S_i$, we can choose $n^*$ vertices in $S_i$ that will contribute to the tree $U_j$, with which we are coupling the current tree $T_j$. If we do discover a vertex of degree more than $D$ in $G_{n,p}$, then for we complete the construction of the trees $U_1, \ldots, U_{n'}$ without embedding the remaining $U_j$ into the trees $T_j$ (to make sure that the $U_1, \ldots, U_{n'}$ are well defined).

Writing $X_v$ for the number of $T$-extensions of the vertex $v$ in $G_{n,p}$, by the properties of the coupling constructed above it follows that

$$\mathbb{P} \left( \max_{v \in [n]} X_v < k \right) \leq \mathbb{P} \left( \max_{i \in [n']} f_T(T_i) < k, \Delta \leq D \right) + \mathbb{P}(\Delta > D)$$

$$\leq \mathbb{P} \left( \max_{i \in [n']} f_T(U_i) < k \right) + o(1),$$

where for the last inequality we recalled that whp $\Delta \leq D$. Let $\pi_n := \mathbb{P}(f_T(T_{n^*,p}) \geq k)$. Since $D = \Theta(\log n)$, definition of $n'$ and assumption (49) imply $\pi_n n' \gg 1$. Since the trees $U_1, \ldots, U_{n'}$ are independent copies of $T_{n^*,p,h}$, and $f_T(T_{n^*,p,h}) = f_T(T_{n^*,p})$, it follows that

$$\mathbb{P} \left( \max_{i \in [n']} f_T(U_i) < k \right) = (1-\pi_n)^{n'} \leq e^{-\pi_n n'} \rightarrow 0,$$

which together with (51) and the definition of $X_v$ completes the proof of Lemma 13. □

**Proof of Lemma 14.** Given a vertex $v \in [n]$, let $T_v$ denote the $h$-neighborhood tree of $v$ in $G_{n,p}$, as defined before the proof of Lemma 13. By construction of $T_v$ (and the trivial bound $|S_i| \leq n$) there is a coupling of $T_{n,p}$ and $T_v$ so that $T_v$ is a subtree of $T_{n,p}$ (with the same root), which in particular ensures that $f_T(T_v) \leq f_T(T_{n,p})$. Assumption (50) and a union bound over $n$ vertices $v \in [n]$ implies that

$$\max_{v \in [n]} f_T(T_v) \leq k \text{ whp.}$$

Recall that $X_v$ is the number of $T$-extensions of $v$ in $G_{n,p}$. Intuitively, we should typically have $X_v \approx f_T(T_v)$ since the neighborhood around $v$ looks like $T_v$ plus a few *surplus* edges, i.e., edges in $G_{n,p} \setminus T_v$ spanned by $V(T_v)$. To exploit this intuition, let $\mathcal{B}$ denote the ‘bad’ event that $G_{n,p}$ contains a cycle $C$ of length at most $L := 2h$ which is within graph distance at most $L$ of a vertex with degree at least $(1+c)pn$. We claim that

$$\neg \mathcal{B} \Rightarrow X_v \leq \max\{f_T(T_v), k\} \text{ for all } v \in [n].$$

(53)
To prove (53), we fix \( v \in [n] \). When \( T_v \) has no surplus edges, then \( X_v = f_T(T_v) \), establishing (53). We henceforth consider the case when \( T_v \) has at least one surplus edge, which means that vertex \( v \) is within distance \( h \) to a cycle of length at most 2\( h \). The event \( \neg \mathcal{B} \) implies that all vertices within distance \( 2h \) of \( v \) have degree at most \((1 + c)pn\). Note that we can define a \( T \)-extension of \( v \) by iteratively mapping the \( v_T - 1 = e_T \) non-root vertices in suitable order, so that each vertex is adjacent to the root vertex \( v \) or a vertex already mapped to. Since \( T \) has height \( h \), it follows that we have at most \((1 + c)pn\) choices for each non-root vertex. Hence \( X_v \leq ((1 + c)pn)^e_T \leq k \), establishing (53).

To complete the proof of Lemma 14, in view of (52) and (53) it suffices to show that

\[
\mathbb{P}(\mathcal{B}) = o(1). \quad (54)
\]

Note that if the event \( \mathcal{B} \) occurs, then \( G_{n,p} \) contains a cycle \( C \) of length \( 3 \leq \ell \leq L \) and a path \( P \) of length \( 0 \leq i \leq L \) connecting \( C \) to a vertex \( v \) of degree at least \((1 + c)pn\), where \( C \) and \( P \) share exactly one vertex (so \( v \) is contained in \( C \) when \( i = 0 \)). Note that \( v \) is uniquely determined by the choice of \( C \) and \( P \), and that \( v \) has at most \(|C \cup P| = \ell + i \leq 2L \) neighbors in the subgraph \( C \cup P \). Using a standard union bound argument that takes all such cycles \( C \) and connecting paths \( P \) into account, it thus follows that

\[
\mathbb{P}(\mathcal{B}) \leq \sum_{3 \leq \ell \leq L} \sum_{0 \leq i \leq L} n^\ell \cdot \ell n^i \cdot \ell^{i+1} \cdot \mathbb{P}(\text{Bin}(n - (\ell + i), p) \geq (1 + c)pn - 2L). \quad (55)
\]

Recall that \( c \in (0,1] \) is fixed. Using stochastic domination together with \( pn \to \infty \), by the Chernoff bound (20) it follows (for all large enough \( n \)) that

\[
\Pi_{\ell,i} \leq \mathbb{P}(\text{Bin}(n, p) \geq (1 + c/2)pn) \leq e^{-\phi(c/2)pn}.
\]

Since \( c, L > 0 \) are constants, using again \( pn \to \infty \) it readily follows that

\[
\mathbb{P}(\mathcal{B}) \leq L(L + 1) \cdot L \cdot (pn)^{2L} \cdot e^{-\phi(c/2)pn} = o(1),
\]

completing the proof of (54) and thus Lemma 14, as discussed. \( \square \)

5. Paths: the sparse case

In this section we deal with extension counts of rooted paths in the ‘sparse’ case \( 1 \ll pn \ll \log n \). In particular, we state and prove Theorem 15 and Proposition 16 below, which generalize Theorem 5 to any \( m \)-edge path \( P_m \) rooted at an endpoint.

We start by recursively defining \( k_m = k_m(n, \lambda) \) for positive integers \( n \) and \( \lambda \in (0, \log n) \):

\[
k_m(n, \lambda) := \begin{cases} 
\log n & \text{if } m = 1, \\
\log \frac{\log n}{\log 1 + \log n \cdot \frac{\log n}{\log \lambda}} & \text{if } m \geq 2.
\end{cases} \quad (55)
\]

**Theorem 15** (Maximum for paths, sparse case). Fix \( T = P_m \), with \( m \geq 1 \). Set \( \lambda := pn \). If \( 1 \ll \lambda \ll \log n \), then

\[
\frac{M_n}{k_m(n, \lambda)} \xrightarrow{p} 1. \quad (56)
\]

We will also show that the \( k_m \) satisfies the following asymptotics. For any integer \( i \geq 1 \), we denote by \( \log^{(i)} \) the natural logarithm, iterated \( i \) times (in particular \( \log^{(1)} = \log \)).
Proposition 16. For $m \geq 1$, the function $k_m = k_m(n, \lambda)$ defined in (55) is increasing in $\lambda$, and satisfies

\[
k_m(n, \lambda) \sim \begin{cases} \frac{\lambda^{m-1} \log n}{\log(\log n)} & \text{if } 1 \leq \lambda \ll \log^{(m)} n, \\ \frac{\lambda^{m-1} \log n}{\log(1+\alpha)} & \text{if } \lambda \sim (\log^{(i)} n)/C \text{ for constant } C, \text{ with } 2 \leq i \leq m. \end{cases}
\] (57)

Proof of Theorem 5. Combine Theorem 15 and Proposition 16 for $m = 2$. □

Proof of Corollary 6. The upper bound follows trivially from Theorem 5, using the simple inequality $X_{T_{1,1},v} \leq (X_{T_{1,1},v})^a$.

For the lower bound we consider the number $Y_v := (X_{T_{1,1},v})^a - X_{T_{1,1},v}$ of $a$-tuples of $T_{1,1}$-extensions that are not vertex disjoint outside of $v$. Using Theorem 5 it suffices to show that $Y_v = o(\alpha_n^a)$ whp. Note that, having chosen an $(a-1)$-tuple of $T_{1,1}$-extensions, the number of ways to choose one more $T_{1,1}$-extension that overlaps with one of them, is clearly at most a constant multiple of the maximum degree $D$. Therefore we have, whp,

\[
Y_v = O(X_{T_{1,1},v}^{a-1}) = O(\alpha_n^{a-1} D)
\]

for any $v \in [n]$. Simple calculus shows that $D \ll \alpha_n$, completing the proof of (16). □

Applying the inequality $\log(1 + x) \leq x$ to the denominator in (55), it follows that

\[
k_m \geq \lambda k_{m-1} \quad \text{for } m \geq 2,
\] (58)

which has a natural interpretation: for the vertex which has $k_{m-1}$ many $P_{m-1}$-extensions, we intuitively expect that on average each $P_{m-1}$-extension extends to about $\lambda = pn$ many $P_m$-extensions. The asymptotics (57) imply that $k_m \sim \lambda k_{m-1}$ if and only if $\lambda \gg \log^{(m)} n$, and thus the message of Theorem 15 is that in this regime the optimal strategy to obtain many $P_m$-extensions is ‘inherited’ from $P_{m-1}$, while in the complementing case $\lambda = O(\log^{(m)} n)$ we have a better strategy, genuinely using the length $m$ of the path.

The proof of Theorem 15 relies on a reduction of the $P_m$-extension counts in $G_{n,p}$ to the size of level $m$ in the Galton–Watson tree $T_{n,p}$ with binomial offspring distribution. The core Lemma 17 in Section 5.1 determines at which value the tails of the size of level $m$ in $T_{n,p}$ changes from being much smaller than $1/n$ to much larger than $1/n$. In Section 5.2 we then use the auxiliary results from Section 4 to deduce Theorem 15 from Lemma 17 in a straightforward way (and also prove the asymptotic formula Proposition 16).

5.1. Random tree: tails of level sizes. Recall that $T_{n,p}$ denotes the Galton–Watson tree with binomial offspring distribution $\text{Bin}(n, p)$. Given an integer $m \geq 1$, let $Z_m$ denote the number of vertices of $T_{n,p}$ of depth $m$ (also called the size of level $m$).

Lemma 17. Fix $m \geq 1$. Let $\lambda := pn$, and assume that $1 \leq \lambda \ll \log n$. Then, for any constant $\alpha > 0$,

\[
P(Z_m \geq \alpha k_m(n, \lambda)) = n^{-\alpha+o(1)}.
\] (59)

Proof. Note that by Proposition 16, using monotonicity and (57) it follows that

\[
k_m(n, \lambda) \geq k_m(n, 1) \sim \frac{\log n}{\log^{(m+1)} n} \gg 1 \quad \text{for any fixed integer } m \geq 1.
\] (60)

A certain optimization problem will play a key role in our upcoming large deviation type arguments. To prepare for this, we now prove that, for any $\rho > 0$ and $\beta \in (0, 1]$,

\[
\beta \left(1 + \frac{1}{\rho} \phi \left(\frac{\rho}{\beta \log(1 + \rho)} - 1\right)\right) \geq 1 \quad \text{with equality for } \beta = \beta_p := \frac{\rho}{\rho + \phi(\rho)},
\] (61)
where $\phi(x)$ is as defined in (19). To prove this, we treat $\rho$ as fixed and use change of variable $x := \frac{\rho}{\beta \log(1+\rho)} - 1$. Note that $x \geq \frac{\rho}{\log(1+\rho)} - 1 > 0$. Since $\beta = \frac{\rho}{(1+x) \log(1+\rho)}$, we can rewrite the inequality in (61) as

$$\frac{\rho + \phi(x)}{(1 + x) \log(1 + \rho)} \geq 1. \quad (62)$$

Denoting the left-hand side of (62) by $f_\rho(x)$, straightforward calculus gives

$$f'_\rho(x) = \frac{(1 + x) \log(1 + x) - \phi(x) - \rho}{(1 + x)^2 \log(1 + \rho)},$$

which implies that on $(0, \infty)$ the function $f_\rho$ is minimized at $x = \rho$, with value $f_\rho(\rho) = \frac{\phi(\rho) + \rho}{(1 + \rho) \log(1 + \rho)} = 1$. This implies (62) and thus (61), since $x = \rho$ corresponds to $\beta = \beta_\rho$.

We are now ready to prove estimate (59) using induction on $m$. In what follows we use the shorthand $k_i = k_i(n, \lambda)$ to avoid clutter. In the base case $m = 1$ we simply have $Z_1 \sim \text{Bin}(n, p)$ and $k_1 = D(n, p)$, so that (59) with $m = 1$ follows from (23) and (24). For the induction step we henceforth assume $m \geq 2$. Fix any constant $\varepsilon \in (0, 1)$ such that $\alpha / \varepsilon$ is an integer. Note that if we condition on $Z_{m-1} \in [x, y)$ for some numbers $x, y$, then the random variable $Z_m$ is stochastically dominated by $\text{Bin}([y] n, p)$. Therefore

$$\mathbb{P}(Z_m \geq \alpha k_m) \leq \mathbb{P}(Z_{m-1} \geq \alpha k_{m-1})$$

$$+ \sum_{1 \leq i \leq \alpha / \varepsilon} \mathbb{P}(Z_{m-1} \in [(i - 1) \varepsilon k_{m-1}, i \varepsilon k_{m-1})] \mathbb{P}(\text{Bin}([i \varepsilon k_{m-1}] n, p) \geq \alpha k_m).$$

Note that by (58) we have $\alpha k_m \geq \alpha k_{m-1} \lambda \geq i \varepsilon k_{m-1} p n$, so the Chernoff bound (20) applies to the binomial tail. By invoking the induction hypothesis for $Z_{m-1}$ (together with the trivial inequality $\mathbb{P}(Z_{m-1} \geq 0) \leq 1 = n^{o(1)}$ in the case $i = 1$), it follows that

$$\mathbb{P}(Z_m \geq \alpha k_m)$$

$$\leq n^{-\alpha + o(1)} + \sum_{1 \leq i \leq \alpha / \varepsilon} n^{-(i-1) \varepsilon + o(1)} \cdot \exp \left( -i \varepsilon k_{m-1} \lambda \phi \left( \frac{\alpha k_m}{i \varepsilon k_{m-1} \lambda} - 1 \right) \right), \quad (63)$$

where we omitted the rounding to integers because the function $\mu \mapsto \mu \phi(x/\mu - 1) = x \log(x/\mu) - x + \mu$ is decreasing on $(0, x]$ (as can be seen by calculating the derivative). Recall the definition of $k_m$ from (55). The logarithm of the $i$th term in (63) is

$$-i \varepsilon \log n \left( 1 + \frac{k_{m-1} \lambda}{\log n} \phi \left( \frac{\alpha \log n}{i \varepsilon k_{m-1} \log (1 + \frac{\log n}{\lambda k_{m-1}})} - 1 \right) \right) + (\varepsilon + o(1)) \log n$$

$$\rho_n := \frac{\log n}{\lambda k_{m-1}}, \beta := i \varepsilon / \alpha$$

$$= - (\alpha \log n) \cdot \beta \left( 1 + \frac{1}{\rho_n} \phi \left( \frac{\rho_n}{\beta \log (1 + \rho_n)} - 1 \right) \right) + (\varepsilon + o(1)) \log n$$

$$\leq - (\alpha \log n) + (\varepsilon + o(1)) \log n. \quad (61)$$

Since $\varepsilon, \alpha$ are constants, using (63) we obtain that

$$\mathbb{P}(Z_m \geq \alpha k_m) \leq n^{-\alpha + o(1)} + \sum_{1 \leq i \leq \alpha / \varepsilon} n^{-\alpha + \varepsilon + o(1)} \leq (1 + \alpha / \varepsilon) \cdot n^{-\alpha + \varepsilon + o(1)} \leq n^{-\alpha + \varepsilon + o(1)},$$

which establishes the upper bound of (59) since $\varepsilon > 0$ was arbitrary.

To complete the proof of the induction step $m \geq 2$, it remains to establish the lower bound of (59). Define $\alpha_n$ so that $\alpha_n k_m = \lceil \alpha k_m \rceil$. Since $k_m \to \infty$ by (60), we have

$$\alpha_n \to \alpha. \quad (64)$$
With an eye on the form of (61) and (63), with foresight we set

\[ \rho_n := \frac{\log n}{\lambda k_{m-1}} \quad \text{and} \quad \gamma_n := \alpha_n \beta_{\rho_n}, \quad (65) \]

where \( \beta_{\rho_n} = \rho_n / (\rho_n + \phi(\rho_n)) \) is as in (61).

Since \( \gamma_n \) is (in general) not a constant, some care is needed when applying the induction hypothesis. We claim that we still have

\[ \mathbb{P}(Z_{m-1} \geq \gamma_n k_{m-1}) \geq n^{-\gamma_n + o(1)}, \quad (66) \]

to see this, note that \( \lim \sup_{n \to \infty} \gamma_n \leq \alpha \) (because \( \beta_{\rho_n} < 1 \) and \( \alpha_n \to \alpha \)). By the subsequence principle it suffices to prove (66) under the assumption that \( \gamma_n \to \gamma \in [0, \alpha] \). For any constant \( \varepsilon > 0 \), the induction hypothesis then implies that, for sufficiently large \( n \),

\[ \mathbb{P}(Z_{m-1} \geq \gamma_n k_{m-1}) \geq \mathbb{P}(Z_{m-1} \geq (\gamma + \varepsilon) k_{m-1}) \geq n^{-(\gamma + \varepsilon) + o(1)} = n^{-(\gamma_n + \varepsilon) + o(1)}. \]

Since \( \varepsilon > 0 \) was arbitrary, this implies the claimed lower bound (66).

Conditioning on \( Z_{m-1} \geq \gamma_n k_{m-1} \) we have that the \( Z_m \) stochastically dominates a random variable \( X \sim \text{Bin}(\lceil \gamma_n k_{m-1} \rceil, p) \). We claim that

\[ \mathbb{P}(X \geq \alpha_n k_m) \geq n^{o(1)} \exp \left( -\gamma_n k_{m-1} \lambda \phi \left( \frac{\alpha_n k_m}{\gamma_n k_{m-1} \lambda} - 1 \right) \right), \quad (67) \]

The rounding of \( \gamma_n k_{m-1} \) causes a technical issue which we overcome by considering two cases: (i) \( \mathbb{E}X = \lceil \gamma_n k_{m-1} \rceil \lambda \leq \alpha_n k_m \) and (ii) \( \mathbb{E}X = \lceil \gamma_n k_{m-1} \rceil \lambda > \alpha_n k_m \). Recalling that \( \lambda \ll \log n \), an easy argument by induction on \( m \) shows that

\[ k_i \ll (\log n)^i, \quad i = 1, 2, \ldots. \quad (68) \]

In the case (i), recalling that \( \alpha_n k_m \) is an integer, Proposition 7 implies

\[ \log \mathbb{P}(X \geq \alpha_n k_m) \geq -\left\lceil \gamma_n k_{m-1} \lambda \phi \left( \frac{\alpha_n k_m}{\gamma_n k_{m-1} \lambda} - 1 \right) \right\rceil + O(\log \alpha_n k_m) \]

which implies (67) in the case (i). In the somewhat degenerate case (ii), we recall that \( \alpha_n k_m \) is an integer, and therefore is at most \( \lceil \mathbb{E}X \rceil \), which, as is well known, is at most the median of \( X \). Hence

\[ \mathbb{P}(X \geq \alpha_n k_m) \geq \mathbb{P}(X \geq \lceil \mathbb{E}X \rceil) \geq 1/2 = n^{o(1)}, \]

which is at least the right-hand side of (67), because \( \phi \) takes only nonnegative values.

Finally we are ready to infer the lower bound of (59) as follows:

\[ \log \mathbb{P}(Z_m \geq \alpha k_m) = \log \mathbb{P}(Z_m \geq \alpha_n k_m) \]

\[ = \log \left( \mathbb{P}(Z_{m-1} \geq \gamma_n k_{m-1}) \cdot \mathbb{P}(X \geq \alpha_n k_m) \right) \]

\[ \geq -\gamma_n \log n + o(\log n) - \gamma_n k_{m-1} \lambda \phi \left( \frac{\alpha_n k_m}{\gamma_n k_{m-1} \lambda} - 1 \right) \]

\[ = -\gamma_n \log n \left( 1 + \frac{\lambda k_{m-1}}{\log n} \phi \left( \frac{\alpha_n \log n}{\gamma_n \lambda k_{m-1} \log (1 + \frac{\log n}{\lambda k_{m-1}})} - 1 \right) \right) + o(\log n) \]

\[ = -\alpha_n \log n \beta_{\rho_n} \left( 1 + \frac{1}{\rho_n} \phi \left( \frac{\rho_n}{\beta_{\rho_n} \log (1 + \rho_n)} - 1 \right) \right) + o(\log n) \]

\[ = -\alpha_n \log n + o(\log n) \]

\[ = -\gamma_n \log n \beta_{\rho_n} \left( 1 + \frac{1}{\rho_n} \phi \left( \frac{\rho_n}{\beta_{\rho_n} \log (1 + \rho_n)} - 1 \right) \right) + o(\log n) \]

\[ = -\gamma_n \log n - (\alpha + o(1)) \log n, \]
which completes the induction step and thus the proof of Lemma 17. \qed

5.2. Proof of main result for paths. In this section we first deduce Theorem 15 from Lemma 17 and Proposition 16, and then give the deferred proof of Proposition 16.

Proof of Theorem 15. It suffices to show that, for any constant $\varepsilon \in (0, 1)$, we whp have
\[
(1 - \varepsilon)k_m(n, \lambda) \leq M_n \leq (1 + \varepsilon)k_m(n, \lambda). \tag{69}
\]

We start with the lower bound in (69). Let $n^* = n^*(n)$ be the sequence from Lemma 13. Since $T = P_m$ is the $m$-edge path, the random variable $f_T(T_{n^*, p})$ is exactly the size of the $m$-th level of $T_{n^*, p}$. Lemma 17 with $n = n^*$ implies, for any constant $\varepsilon \in (0, 1)$, that
\[
\Pr(f_T(T_{n^*, p}) \geq (1 - \varepsilon/2)k_m(n^*, p n^*)) \geq (n^*)^{-(1-\varepsilon/2)+o(1)} \gg (\log n)^{m+1}/n.
\]

Using the estimate (57), it is straightforward to check that $1 \ll pn = \lambda \ll n$ and $n^* \sim n$ imply $k_m(n^*, pn^*) \sim k_m(n, \lambda)$. Applying Lemma 13 it thus follows that, whp,
\[
M_n \geq (1 - \varepsilon/2)k_m(n^*, pn^*) \geq (1 - \varepsilon)k_m(n, \lambda),
\]
which establishes the lower bound in (69).

We now turn to the upper bound in (69). Since the random variable $f_T(T_{n, p})$ is the size of the $m$-th level of $T_{n, p}$, Lemma 17 implies for any constant $\varepsilon \in (0, 1)$ that
\[
\Pr(f_T(T_{n, p}) \geq (1 + \varepsilon)k_m(n, \lambda)) \leq n^{-(1+\varepsilon)+o(1)} \ll 1/n.
\]

The assumption $\lambda \ll \log n$ implies $k_1/\lambda = \log n / (\log \log n) \to \infty$, therefore using inequality (58) we readily infer that $k_m \geq \lambda^{m-1}k_1 \gg \lambda^m = (pn)^{\tau}$. Applying Lemma 14 it thus follows that, whp, $M_n \leq (1 + \varepsilon)k_m(n, \lambda)$, which establishes the upper bound in (69), completing the proof of Theorem 15. \qed

Proof of Proposition 16. Monotonicity of $k_m$ follows by induction: this is easy for $m = 1$, and monotonicity of $k_m$ then follows from monotonicity of $k_{m-1}$ and $log$.

To prove (57) we also use induction on $m$. The base case $m = 1$ is immediate from the definition of $k_1$. In the induction step $m \geq 2$ we employ a case distinction. If $1 \leq \lambda = O(\log^m n)$ holds, then by invoking the induction hypothesis (using $1 \leq \lambda = O(\log^m n) \ll \log^(m-1) n$) it follows that
\[
\frac{\log n}{\lambda k_{m-1}(n, \lambda)} \sim \frac{\log(\log^m n)}{\lambda} \sim \frac{\log^m n}{\lambda},
\]
which in view of (55) yields the claimed asymptotics (57) in the cases $1 \ll \lambda \ll \log^m n$ and $\lambda \sim (\log^m n)/C$. If $\lambda \gg \log^m n$ holds, then by invoking monotonicity and the induction hypothesis (using $1 \ll \log^m n \ll \log^(m-1) n$) it follows that
\[
\lambda k_{m-1}(n, \lambda) \gg (\log^m n) \cdot k_{m-1}(n, \log^m n) \sim \frac{(\log^m n) \log n}{\log(\log^m n)} \sim \log n.
\]

Hence, combining the definition (55) and the asymptotics $\log(1 + x) \sim x$ as $x \to 0$ with the induction hypothesis, it follows that
\[
k_m(n, \lambda) \sim \lambda k_{m-1}(n, \lambda)
\sim \begin{cases} 
\frac{\lambda^{m-1} \log n}{\log(\log^m n)} & \text{if } \log^{(i+1)} n \ll \lambda \ll \log(i) n \text{ with } 1 \leq i \leq m - 1, \\
\frac{\lambda^{m-1} \log n}{\log(\log^m n)} & \text{if } \lambda \sim (\log^i n)/C \text{ with } 2 \leq i \leq m - 1,
\end{cases}
\]

which yields the claimed asymptotics (57) in the remaining cases. \qed
6. Spherically symmetric trees: the sparse case

In this section we deal with extension counts of spherically symmetric trees $T_{a,b}$ (defined in Section 1.2) in the ‘sparse’ case $1 \ll pn \ll \log n$. In particular, we state and prove Theorem 18 below, which generalizes Theorem 4 by including the intermediate regime $pn \gg (\log n/\log \log n)^{1-1/b}$ not covered by (12)–(13), in which case the optimal strategy interpolates between the two distinct strategies leading to (8) and (11).

We start by recalling that the maximum degree $\Delta$ of $G_{n,p}$ is concentrated around $D = D(n,p)$ defined in (7). It is routine to check that the condition $pn \gg (\log n/\log \log n)^{1-1/b}$ is equivalent to $D^{b-1} \approx (pn)^b$, so in Theorem 18 below we do not lose generality by assuming that $D^{b-1}/(pn)^b$ converges to some limit $L \in [0, \infty]$.

**Theorem 18** (Maximum for trees $T_{a,b}$, sparse case). Fix $T = T_{a,b}$, with $a \geq 1$ and $b \geq 2$. Suppose that $1 \ll pn \ll \log n$ and $D^{b-1}/(pn)^b \to L \in [0, \infty]$. Then

$$\frac{M_n}{\alpha_n} \xrightarrow{p} 1,$$

(70)

where, for a suitable constant $C_{a,b} \in [1, \infty)$, we have

$$\alpha_n := \begin{cases} [D(pn)^b]_a, & \text{if } L \in [0, 1], \\ [D(pn)^b] \sup_{(x_0, \ldots, x_k) \in \Lambda} F_L(x_0, \ldots, x_k), & \text{if } L \in (1, C_{a,b}), \\ D^{ab} \sup_{(x_1, \ldots, x_k) \in \Lambda} f_{a,b}(x_1, \ldots, x_k), & \text{if } L \in [C_{a,b}, \infty], \end{cases}$$

(71)

where the set $\Lambda$ is defined as in (9), the function $f_{a,b}$ is defined as in (10), and

$$F_L(x_0, \cdots, x_k) := \sum_{m=0}^{a} \binom{a}{m} L^m x_0^{a-m} f_{m,b}(x_1, \cdots, x_k).$$

(72)

**Remark 4.** In the latter two cases the (asymptotic) extreme value $\alpha_n$ depends on the optimal value of an optimization problem. This problem, especially in the final case, appears quite elementary, as it simply depends on optimizing multivariate polynomials over simple convex sets. However, we do not know how to find the optimum in general, and it would be interesting to know these optimum values. It is quite possible that the optimum is always achieved by constant vectors $(1/k, \ldots, 1/k)$, for some $k$. The optimization problem is discussed further in the appendix to the arxiv version of this paper.

**Proof of Theorem 4.** Using the definition (7) of $D$, note that $pn \gg (\log n/\log \log n)^{1-1/b}$ implies $D^{b-1}/(pn)^b \to 0$, and that $pn \ll (\log n/\log \log n)^{1-1/b}$ implies $D^{b-1}/(pn)^b \to \infty$. In view of this, now (70)–(71) of Theorem 18 imply Theorem 4. \hfill \square

Let us provide heuristics for the denominator $\alpha_n$ in Theorem 18 in the case $L \in (0, \infty)$. We will see that a combined strategy gives a lower bound that interpolates between the bounds (8) and (11) discussed in the introduction. Introducing an extra variable $x_0$, consider a vector $(x_0, x_1, \cdots, x_k)$ of positive numbers with $x_0 + \cdots + x_k = 1$. It can be shown that with probability $n^{-x_0+o(1)}$ a vertex has $x_0D$ neighbors, most of which have degree $(1+o(1))pn$. Moreover, with probability $n^{-(x_1+\cdots+x_k)+o(1)}$ a vertex has neighbors with their degrees at least $x_1D, \ldots, x_kD$. The probability to have both types of neighbors is therefore $n^{-(x_0+\cdots+x_k)+o(1)} \approx n^{-1}$. Classifying the extensions according to the number $m$ of children of the root that are mapped to vertices of degrees $x_1D, \cdots, x_kD$ (and the remaining $a - m$ to the $x_0D$ vertices of degree $pn$), we thus ought to be able to find a
vertex \( v \) such that, whp,

\[
X_v \geq (1 + o(1)) \sum_{m=0}^{a} \binom{a}{m} (x_0 D)^{a-m} (pn)^{(a-m)b} f_{m,b}(x_1, \ldots, x_k) D^{mb}
\]

\[
\sim [D(pn)^b]^a \sum_{m=0}^{a} \binom{a}{m} \left( \frac{D^{b-1}}{(pn)^b} \right)^m x_0^{a-m} f_{m,b}(x_1, \ldots, x_k)
\]

\[
\sim [D(pn)^b]^a F_L(x_0, \ldots, x_k).
\]

Taking the supremum over all vectors \((x_0, x_1, \cdots)\) with sum at most one, we thus ought to be able to find a vertex \( v \) such that, whp,

\[
X_v \geq (1 + o(1))[D(pn)^b]^a \sup_{(x_0, \ldots, x_k) \in \Lambda} F_L(x_0, \ldots, x_k).
\]

The following proposition claims that supremum in (73) can be simplified if the limit \( L \) is sufficiently small or large. It would be interesting to know if equality in (74) actually holds for all \( L \), since then Theorem 18 would take a simpler form.

**Proposition 19.** Let \( a \geq 1 \) and \( b \geq 2 \) be integers. For every \( L \in [0, \infty) \) we have

\[
\sup_{(x_0, \ldots, x_k) \in \Lambda} F_L(x_0, \ldots, x_k) \geq \max \{ 1, L^a \sup_{(x_1, \ldots, x_k) \in \Lambda} f_{a,b}(x_1, \ldots, x_k) \}.
\]

Moreover, there is a constant \( C_{a,b} \geq 1 \) such that

\[
\sup_{(x_0, \ldots, x_k) \in \Lambda} F_L(x_0, \ldots, x_k) =
\begin{cases} 
1 & \text{if } L \in [0, 1], \\
L^a \sup_{(x_1, \ldots, x_k) \in \Lambda} f_{a,b}(x_1, \ldots, x_k) & \text{if } L \in [C_{a,b}, \infty).
\end{cases}
\]

**Proof.** Inequality (74) follows by setting the first argument of \( F_L \) to \( x_0 = 1 \), or setting \( x_0 = 0 \) and optimizing over \((x_1, \cdots) \in \Lambda \).

To prove (75), writing

\[
\lambda_m := \sup_{(x_1, \ldots, x_k) \in \Lambda} f_{m,b}(x_1, \ldots, x_k),
\]

we claim that, for every \((x_0, \ldots, x_k) \in \Lambda\), we have \( f_{m,b}(x_1, \ldots, x_k) \leq (1 - x_0)^m \lambda_m \). This claim is trivial when \( x_0 = 1 \) and when \( x_0 < 1 \) it follows by noting that

\[
f_{m,b}(x_1, \cdots, x_k) = (1 - x_0)^m f_{m,b}\left(\frac{x_1}{1 - x_0}, \ldots, \frac{x_k}{1 - x_0}\right) \leq (1 - x_0)^m \lambda_m \leq (1 - x_0)^m \lambda_m.
\]

Hence, fixing \( x_0 \) and bounding each term of \( F_L \) separately, we get that

\[
\sup_{(x_0, \ldots, x_k) \in \Lambda} F_L(x_0, \ldots, x_k) \leq \max_{x_0 \in [0,1]} \sum_{m=0}^{a} \binom{a}{m} x_0^{a-m} (1 - x_0)^m L^m \lambda_m
\]

\[
\leq \max \{ L^a \lambda_m : m = 0, \cdots, a \},
\]

where the last inequality follows because the sum is a convex combination of numbers \( L^m \lambda_m \). It is easy to show that \( \lambda_0 = \lambda_1 = 1 \) and that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_a > 0 \). Hence for \( L \leq 1 \), the number \( L^m \lambda_m \) is maximized by \( m = 0 \), together with lower bound (74) giving \( \sup_{(x_0, \ldots, x_k) \in \Lambda} F_L(x_0, \ldots, x_k) = 1 \), while for sufficiently large \( L \) we have that maximum is \( L^a \lambda_a \), giving the second case in (75). \( \square \)

Our proof of Theorem 18 exploits the auxiliary results from Section 4, which allow us to focus on the number of \( T_{a,b} \)-extensions of the root in \( T_{n,p} \), a Galton–Watson tree with offspring distribution \( \text{Bin}(n, p) \). These extensions may be classified depending on the degrees of the children of the root. In Subsection 6.1 we prove bounds related to high degree vertices, and in Subsection 6.2 we prove bounds related to intermediate degree.
vertices. In Subsection 6.3 we use these bounds to prove the upper bound of Theorem 18 and in Subsection 6.4 we give a self-contained proof of the lower bound of Theorem 18.

To discuss \( T_{n,p} \) throughout the whole section, we introduce the following notation. Let \([n]_p \subseteq [n]\) denote the set of indices of the children of the root (i.e., every element of \([n]\) is included in \([n]_p\) independently with probability \(p\)), and let \(\xi_1, \ldots, \xi_n\) be the independent random variables (also independent from \([n]_p\) with distribution \(\text{Bin}(n, p)\)) so that the degrees of the children of the root are \((\xi_i : i \in [n]_p)\).

### 6.1. High degree neighbors

It may well seem natural to consider vertices of large degree being of degree at least \(\eta D\) for some constant \(\eta \in (0, 1)\), but we find it more convenient to take

\[
\eta := \frac{1}{(\log \log n)^5}.
\] (76)

Lemma 20 is the main result of this subsection: it provides convenient bounds on

\[
H := \sum \{\xi_i : i \in [n]_p, \xi_i \geq \eta D\},
\] (77)

which, in concrete words, is the sum of the degrees of high degree neighbors of the root.

#### Lemma 20

Fix \(\theta, \delta \in (0, 1)\). If \(pn \leq (\log n)\theta\), then

\[\mathbb{P}(H \geq (1 + \delta)D) \ll 1/n,\] (78)

and, for any constant \(x_0 \in (0, 1 + \delta)\), we also have

\[\mathbb{P}(||[n]_p| \geq x_0 D, H \geq (1 - x_0 + \delta)D) \ll 1/n.\] (79)

In the upcoming proof of Lemma 20, the discussed choice (76) of \(\eta\) enables us to use the following simple fact about the tail probabilities of \(\xi \sim \text{Bin}(n, p)\).

#### Fact 21

Fix \(\theta \in (0, 1)\). If \(pn \leq (\log n)^\theta\), then for any \(x \in [\eta/2, 2]\) we have

\[\mathbb{P}(\xi \geq xD) \leq n^{-x(1+o(1))}.\] (80)

**Proof.** Apply inequality (23) of Lemma 8 with \(\alpha = x\). \(\square\)

**Proof of Lemma 20.** We will several times tacitly assume that \(n\) is larger than a suitable constant, which possibly depends on \(\theta\) and \(\delta\). We start with inequality (78). If the event \(H \geq (1 + \delta)D\) occurs, then there exist positive numbers \(y_1, \ldots, y_k\) such that

(i) \(\sum_{j=1}^k y_j \geq 1 + \delta\),

(ii) \(k \leq 4\eta^{-1}\),

(iii) \(y_j \geq \eta\) for all \(j \in [k]\), and

(iv) there exist distinct \(i_1, \ldots, i_k \in [n]_p\) such that \(\xi_{i_j} = y_j D\) for all \(j \in [k]\).

(Note that (ii) holds because choosing \(y_j\)’s minimally with respect to (i), we can assume \(\sum_j y_j \leq 4,\) say.) With foresight, set \(\gamma := \delta \eta/16 = \delta/16(\log \log n)^5\). By rounding down if necessary, there exists a vector \(\mathbf{x} = (x_1, \ldots, x_k)\) of positive numbers such that

(i) \(\sum_{j=1}^k x_j \geq 1 + 3\delta/4\),

(ii) \(k \leq 4\eta^{-1}\),

(iii) \(x_j \geq \eta/2\) and \(x_j\) is a multiple of \(\gamma\) for all \(j \in [k]\), and

(iv) there exists distinct \(i_1, \ldots, i_k \in [n]_p\) such that \(\xi_{i_j} \geq x_j D\) for all \(j \in [k]\).

Denote the event in (iv) by \(E_\mathbf{x}\). As the number of choices of \(\mathbf{x}\) satisfying (i)–(iii) minimally is at most

\[((1 + 3\delta/4)\gamma^{-1})^{4\eta^{-1}} \leq e^{(\log \log n)^5} = n^{o(1)},\]
to establish (78) it thus suffices to prove that \( \mathbb{P}(E_x) \leq n^{-1-\delta/2} \) for all sufficiently large \( n \). Using the union bound, inequality (80), and independence of \( \xi_i \)s, it follows that

\[
\mathbb{P}(E_x) \leq n^k p^k \prod_{j=1}^{k} n^{-x_j(1+o(1))} \leq (\log n)^k \cdot n^{-(1+o(1))} \sum_j x_j
\]

\[
\leq n^{o(1)} \cdot n^{-(1+o(1))(1+3\delta/4)} \leq n^{-1-\delta/2}
\]

for all sufficiently large \( n \), completing the proof of inequality (78).

The proof of inequality (79) is essentially identical. Indeed, this time we have probability at most \( n^{-(1-x_0)-\delta/2} \) that \( H \geq (1 - x_0 + \delta)D \), and (conditional) probability at most \( n^{-x_0+o(1)} \) that the root has at least \( (x_0 + o(1))D \) other neighbors, by (80). Thus the probability of the event is at most \( n^{-1-\delta/2+o(1)} \), which is \( o(n^{-1}) \), as required.

\[\Box\]

6.2. Upper bound on extension counts using intermediate degrees. We next consider the contribution of neighbors of the root with intermediate degree, between \( (1 + \varepsilon)pn \) and \( \eta D \), where \( \varepsilon > 0 \) is fixed and \( \eta \) is as is in (76). Lemma 22 is the main result of this subsection: it will later be used to show that contribution from \( T_{a,b} \)-extensions that use neighbors of intermediate degree is negligible.

**Lemma 22.** Let \( a \geq 1 \) and \( b \geq 2 \) be integers. Fix \( \theta < 1 \). Suppose that \( 1 \leq pn \leq (\log n)^\theta \). For any constants \( \delta, \varepsilon > 0 \), with probability \( 1 - o(1/n) \) the root of \( T_{a,b} \) has at most

\[
\delta \left( \max\{D(pm)^b, D^b\} \right)^a
\]

many \( T_{a,b} \)-extensions in which at least one of the neighbors of the root has intermediate degree, i.e., in the interval \([((1+\varepsilon)pn, \eta D)\).

The upcoming proof of Lemma 22 is based on the following two facts (Facts 23 and 24).

**Fact 23.** Suppose that \( 1 \leq pn \leq \log n \). For every \( \varepsilon > 0 \) there exists \( C > 0 \) such that

\[
\mathbb{P}\left( \left| \{i \in [n] : \xi_i \geq (1 + \varepsilon)pn\} \right| \geq C(pm)^{-1} \log n \right) = O(n^{-2}).
\]

**Proof.** Writing \( I_\varepsilon := \{i \in [n] : \xi_i \geq (1 + \varepsilon)pn\} \), our goal is to prove that

\[
\mathbb{P}\left( \left| [n]_p \cap I_\varepsilon \right| \geq C(pm)^{-1} \log n \right) \leq n^{-2}
\]

for some constant \( C > 0 \). By the Chernoff bound (20) there is a constant \( c = c(\varepsilon) > 0 \) such that \( \mathbb{P}(i \in I_\varepsilon) \leq e^{-cpn} \). Write \( s := [C(pm)^{-1} \log n] \). Using the independence of \( \xi_i \)s and elements of \([n]_p\), it follows that

\[
\mathbb{P}\left( \left| [n]_p \cap I_\varepsilon \right| \geq C(pm)^{-1} \log n \right) \leq \left( \frac{n}{s} \right) p^s e^{-cpn} s
\]

\[
\leq \left( \frac{pm_{1-cpn}}{s} \right)^s e^{-cpns/2} \leq n^{-2},
\]

which completes the proof of Fact 23, as discussed.

\[\Box\]

**Fact 24.** If \( 1 \leq pn \leq \log n \), then

\[
\max_{d \geq 64pn} \mathbb{P}(E_d) = O(n^{-2}),
\]

where \( E_d := \{|i \in [n] : \xi_i \geq d| \geq d^{-1} \log n\} \) for any integer \( d \geq 1 \).
Proof. Let $d = d(n) \geq 64pn$ be the integer which maximizes the probability $\mathbb{P}(\mathcal{E}_d)$. Writing $I_d := \{i \in [n] : \xi_i \geq d\}$, our goal is to prove that
$$\mathbb{P}(|[n]_p \cap I_d| \geq d^{-1} \log n) = O(n^{-2}).$$
As $d \geq 64pn$, it follows from the Chernoff bound (21) that
$$\mathbb{P}(i \in I_d) = \mathbb{P}(\xi_i \geq d) \leq \exp(-d \log(d/epn)) \leq e^{-3d}.$$Write $s := [d^{-1} \log n]$. Using independence of $\xi_i$s and elements of $[n]_p$, it follows that
$$\mathbb{P}(|[n]_p \cap I_d| \geq d^{-1} \log n) \leq \binom{n}{s} p^s (e^{-3d})^s$$
$$\leq \left(\frac{pne^{-3d}}{s}\right)^s \leq \exp(-2ds) \leq n^{-2},$$
which completes the proof of Fact 24, as discussed.

Proof of Lemma 22. Suppose that the number of $T_{1,b}$-extensions of the root is $t_b$, and the number of $T_{1,b}$-extensions of the root in which the neighbor of the root has intermediate degree is $t'_b$. We claim that for some constant $C_b$ with probability $1 - o(1/n)$
$$t_b \leq C_b \max\{D(pm)^b, D^b\},$$
and with probability $1 - o(1/n)$
$$t'_b \leq \delta D^b.$$Since the number of $T_{a,b}$-extensions that we consider is at most $at'_b t_b^{a-1}$, bounds (82) and (83) will imply the bound (81) up to adjusting the value of $\delta$.

We start with the proof of (82). For this note that the number of $T_{1,b}$-extensions of the root is at most $\sum_{i \in [n]_p} \xi_i^b$. We bound contributions to this sum from degrees of different sizes. All the bounds stated hold with probability $1 - o(n^{-1})$, and so (82) will follow by a union bound.

We first consider small degrees, at most $64pn$. By (80) we may assume the root has degree at most $2D$, and so the contribution of such neighbors is at most $2(64pn)^b D(pm)^b$.

For degrees in the interval $[d, 2d]$, with $64pn \leq d < \eta D$, we use Fact 24. This bounds their contribution by $(d^{-1} \log n)(2d)^b = 2^b (\log n)D^b$. We may then sum this over intervals of the form $[2^i pn, 2^{i+1} pm)$ that cover the interval $[64pn, \eta D]$. Since the number of intervals is at most $log_2(D/pm) = O(\log \log n)$ and each inequality holds with probability $1 - O(n^{-2})$, we have that with probability $1 - o(n^{-1})$ the total contribution is at most
$$2^b (\log n) \sum_{i=6}^{[\log_2(\eta D/pm)]} (2^i pn)^b \leq 2^{b+1} (\log n) \eta D^{b-1} \leq \frac{D^b}{(\log \log n)^2},$$
provided $n$ is sufficiently large (this is one point where we use that $b \geq 2$ holds).

For degrees at least $\eta D$, the required bound follows from Lemma 20 with $\delta = 1$, say. Indeed, if the sum of the degrees of high degree neighbors is at most $2D$, then the sum of $b$th powers of these degrees is at most $2^b D^b$. This completes the proof of (82).

Finally we prove (83). By Fact 23, except with probability $O(n^{-2})$, the number of $T_{1,b}$-extensions using a neighbor of degree $d \in [(1 + \varepsilon) pm, 64pn]$ is at most
$$C \log n \frac{(64pn)^b}{pn} \leq 64C \log n (pm)^{b-1} \leq \frac{\delta}{2} D^b,$$
for all sufficiently large n (again using that b ≥ 2 holds). On the other hand, the contribution to \( t'_b \) of vertices with degree in the interval \([64pn, \eta D]\), was shown in (84) to be \(o(D^b)\). This completes the proof of estimate (83), and thus of Lemma 22.

\[ \square \]

6.3. Proof of the upper bound of Theorem 18. In order to prove the upper bound of Theorem 18, in view of Lemma 14, it essentially (modulo checking that \( \alpha_n \gg \mu_T \)) suffices to prove that, for any constant \( \varepsilon > 0 \), in \( T_{n,p} \), the number of \( T_{a,b} \)-extensions of the root exceeds \((1 + \varepsilon)\alpha_n\) with probability \(o(1/n)\).

Our argument is based on the following two propositions, whose proofs are deferred. Proposition 25 deals with the cases where \( pn \geq (log n)^{1-1/2b} \), which are contained within the case \( D^b/(pn)^{b-1} \to 0 \), i.e., when \( \alpha_n = [D(pm)^b]^a \). Proposition 26 deals with the remaining sparser and more technical cases where \( pn \leq (log n)^{1-1/2b} \).

**Proposition 25.** Let \( a \geq 1 \) and \( b \geq 2 \). Suppose that \((log n)^{1-1/2b} \leq pn \ll log n\). For every constant \( \delta > 0 \), with probability \( 1 - o(1/n) \) the number of \( T_{a,b} \)-extensions of the root of \( T_{n,p} \) is at most \((1 + \delta)[D(pm)^b]^a\).

**Proposition 26.** Let \( \delta > 0 \) and let \( a \geq 1 \) and \( b \geq 2 \). Suppose that \( 1 \leq pn \leq (log n)^{1-1/2b} \). For every constant \( \delta > 0 \), with probability \( 1 - o(1/n) \) the number of \( T_{a,b} \)-extensions of the root of \( T_{n,p} \) is at most

\[
(1 + a^{ab}\delta)^2 \sup_{(x_0, \ldots, x_k) \in (1 + \delta)A} \sum_{m=0}^a \binom{a}{m} (x_0 D)^{a-m}(pn)^{(a-m)b} f_{m,b}(x_1, \ldots, x_k) D^{mb}. \tag{86}
\]

**Proof of the upper bound of Theorem 18.** In view of Lemma 14, Proposition 25 with \( \delta = \varepsilon \) covers the case \((log n)^{1-1/2b} \leq pn \ll log n\), since, as we noted above, \( \alpha_n = [D(pm)^b]^a \) holds, which due to \( D \gg pn \) (which follows from \( pn \ll log n\)) satisfies \( \alpha_n \gg (pn)^{a+ab} = \Theta(\mu_T) \).

Now, for \( 1 \ll pn \leq (log n)^{1-1/2b} \), we use Proposition 26. We recall from the statement of Theorem 18 that we use \( L \) for the limit of \( D^{b-1}/(pn)^b \) as \( n \) tends to infinity (possibly \( L = \infty \)). We defer the choice of \( \delta = \delta(\varepsilon, a, b) > 0 \). By Lemma 14 and Proposition 26 in \( G_{n,p} \), the maximum number of \( T_{a,b} \)-extensions satisfies, whp, the upper bound

\[
M_n \leq (1 + a^{ab}\delta)^2 \sup_{(x_0, \ldots, x_k) \in (1 + \delta)A} \sum_{m=0}^a \binom{a}{m} (x_0 D)^{a-m}(pn)^{(a-m)b} f_{m,b}(x_1, \ldots, x_k) D^{mb} \\
\leq (1 + a^{ab}\delta)^{ab+2} \sup_{(x_0, \ldots, x_k) \in A} \sum_{m=0}^a \binom{a}{m} (x_0 D)^{a-m}(pn)^{(a-m)b} f_{m,b}(x_1, \ldots, x_k) D^{mb},
\]

provided that \( S_n \gg \mu_T = \Theta((pm)^{(b+1)a}) \) holds. We choose \( \delta = \delta(\varepsilon, a, b) > 0 \) small enough such that the factor in front of \( S_n \) is strictly less than \( 1 + \varepsilon \). Hence it remains to check that \( S_n \sim \alpha_n \) and \( S_n \gg (pm)^{a+ab} \), with \( \alpha_n \) as in (71). For this observe that

\[
\frac{S_n}{[D(pm)^b]^a} = \sup_{(x_0, \ldots, x_k) \in A} \binom{a}{m} \frac{D^{b-1}/(pm)^b}{-1}^{m} (x_0)^{a-m} f_{m,b}(x_1, \ldots, x_k), \tag{87}
\]

where, in the case \( L \in [0, \infty) \), the limit of the sum on the right-hand side of (87) equals \( F_L(x_0, \ldots, x_k) \) for \( F_L \) as in (72). In the case \( L \in [0, \infty) \) we thus infer that

\[
S_n \sim [D(pm)^b]^a \sup_{(x_0, \ldots, x_k) \in A} F_L(x_0, \ldots, x_k) \quad \text{if} \ L \in [0, \infty), \tag{88}
\]

where due to \( D \gg pn \) it is easy to see that \( S_n \sim [D(pm)^b]^a \gg (pm)^{(b+1)a} \). Similarly, in the case \( L = \infty \), it is straightforward to see that the main contribution to the right-hand
side of (87) comes from the case \( m = a \), so we infer that
\[
S_n \sim D^{ab} \sup_{(x_1, \ldots, x_k) \in \Lambda} f_{a,b}(x_1, \ldots, x_k) \quad \text{if } L = \infty,
\] (89)
where due to \( D^{b-1}/(pn)^b \rightarrow L = \infty \) and \( D \gg pn \) it is easy to see that \( S_n \sim D^{ab} = (D^b)^a \gg (D(pm))^b \gg (pn)^{(b+1)a} \ll \mu_T \). Combining the estimates (88) and (89) with Proposition 19, it follows that \( S_n \sim \alpha_n \), which completes the proof of the upper bound of Theorem 18.

We now give the deferred proofs of Propositions 25 and 26.

Proof of Proposition 25. Up to adjusting the value of \( \delta \), it clearly suffices to prove the case \( a = 1 \), as the number of \( T_{a,b} \)-extensions is at most the \( a \)th power of the number of \( T_{1,b} \)-extensions.

Inequality (23) of Lemma 8 implies that with probability \( 1 - o(1/n) \) the root has at most \( (1 + \delta^2)D \) neighbors and therefore the number of \( T_{1,b} \)-extensions of the root which use a neighbor of degree at most \( (1 + \delta^2)pm \) is at most
\[
(1 + \delta^2)D (1 + \delta^2)^b (pm)^b \leq (1 + \delta/2) D (pm)^b,
\] (90)
where the inequality holds because we may assume that \( \delta \) is smaller than some constant (dependent only on \( b \)).

Let \( E' \) be the event that the root has more than \( C \log n / pn \) neighbors with degree at least \( (1 + \delta^2)p \), where \( C = C(\delta^2) \) is the constant given by Fact 23. Fact 23 implies that \( \mathbb{P}(E') \ll 1/n \).

Let \( I = \{2i pn : i \geq 6, 2ipn \leq 2D \} \), and for each \( d \in I \) let \( \mathcal{E}_d \) be the event that the root has more than \( \log n / d \) neighbors with degrees at least \( d \). Note that \( \cup_{d \in I} [d, 2d] \) covers all degrees in the interval \([64pn, 2D]\). By Since \( |I| = O(\log n) \), by Fact 24 we have \( \mathbb{P}(\cup_{d \in I} \mathcal{E}_d) \ll 1/n \).

Finally, let \( E'' \) be the event that the root has a neighbor of degree at least \( 2D \). By inequality (23) the expected number of such neighbors is at most \( pn \cdot n^{-2+o(1)} \ll 1/n \), hence \( \mathbb{P}(E'') \ll 1/n \).

We conclude that on the complement of \( E' \cup \bigcup_{d \in I} \mathcal{E}_d \cup E'' \) (i.e., with probability \( 1 - o(1/n) \)) the number of \( T_{1,b} \)-extensions of the root which use a neighbor with degree at least \( (1 + \delta^2)pn \) is at most
\[
\frac{C \log n}{pn} (64pn)^b + \sum_{d \in I} \frac{\log n}{d} (2d)^b \leq 2^{ob} C(pm)^{b-1} \log n + 2^{b+1}(2D)^{b-1} \log n
\]
for sufficiently large \( n \), using \( (\log n)^{1-1/2b} \leq pn \ll \log n \) \leq \( \delta^2 D (pm)^b \),
which together with (90) completes the proof of Proposition 25.

Proof of Proposition 26. We may assume \( \delta \) is at most some small constant that depends on \( a \) and \( b \) only. Set \( \eta = (\log \log n)^{-5} \) as in (76). We recall that a vertex degree is intermediate if it lies in the interval \([1 + \delta^2)pn, \eta D]\) and large if it is at least \( \eta D \). In the rest of the proof by extension we mean a \( T_{a,b} \)-extension of the root of \( T_{a,p} \). Let us call an extension typical if no child of the root of intermediate degree is used. Lemma 22 with \( \varepsilon = \delta^2 \) implies that, with probability \( 1 - o(1/n) \), the total number of extensions which do use a child of the root of intermediate degree is at most
\[
\delta \max\{D^n(pm)^{ab}, D^{ab}\}.
\]
Since \( D^n(pm)^{ab} \) is at most the supremum in (86) (consider \( x_0 = 1, x_1 = x_2 = \ldots = 0 \)), and \( D^{ab} \) is at most \( a^{ab} \) times the supremum (consider \( x_0 = 0, x_1 = \ldots = x_a = 1/a \), with
probability $1 - o(1/n)$ the number of non-typical extensions is at most $a^{ab}\delta$ times the supremum in (86).

What remains to prove is that, with probability $1 - o(1/n)$, there are nonnegative numbers $x_0, x_1, \ldots, x_k$ such that $\sum_i x_i \leq 1 + \delta$ and such that the number of typical extensions is at most

$$(1 + \delta) \sum_{m=0}^{a} \binom{a}{m} (x_0 D)^{a-m} (pn)^{(a-m)b} f_{m,b}(x_1, \ldots, x_k) D^{mb}. \tag{91}$$

Recall from Subsection 6.1 that $[n]_p$ is the set of indices of the children of the root, and $H = \sum \{\xi_i : i \in [n]_p, \xi_i \geq \eta D\}$ is the sum of large degrees among the children of the root. Let $E$ be the event that either $|[n]_p| \geq (1 + \delta/4) D$ or there exists a multiple $x_0$ of $\delta/2$ at most $1 + 3\delta/4$ such that

$$|[n]_p| \in [(x_0 - \delta/2) D, x_0 D] \quad \text{and} \quad H \geq (1 - x_0 + \delta) D. \tag{92}$$

Inequality (80) implies that $\Pr((|n|_p| \geq (1 + \delta/4) D) = o(1/n)$. Moreover, for a particular $x_0$, event (92) has probability at most $o(1/n)$ by Lemma 20. Since there is a constant number of choices of $x_0$, an easy union bound gives us that $\Pr(E) = o(1/n)$.

On the complement of the event $E$, there exists $x_0 \in (0, 1 + \delta]$ such that

$$|[n]_p| \in [(x_0 - \delta/2) D, x_0 D] \quad \text{and} \quad H < (1 - x_0 + \delta) D.$$

In particular, there are nonnegative numbers $x_0, x_1, \ldots, x_k$ with $\sum_{i=0}^{k} x_i \leq 1 + \delta$ such that $|[n]_p| \leq x_0 D$ and the degrees of children of the root of large degree are given by $x_1 D, \ldots, x_k D$. A typical extension may only use neighbors of small degree (at most $(1 + \delta^2)pn$) or large degree. For each $m \in \{0, \ldots, a\}$, let $Y_m$ be the number of typical $T_{a,b}$-extensions in which $m$ neighbors have large degree and $a - m$ have small degree. Considering the information we have on neighbors of the root, we have that

$$Y_m \leq \binom{a}{m} (x_0 D)^{a-m} (1 + \delta^2)^{m} (pn)^{(a-m)b} f_{m,b}(x_1, \ldots, x_k) D^{mb}.$$

Recalling that $\delta$ is at most a small constant, we have that $(1 + \delta^2)^{ab} \leq 1 + \delta$. The required bound (91) now follows as the number of typical $T_{a,b}$-extensions is precisely $\sum_{m=0}^{a} Y_m$. \hfill $\square$

6.4. Proof of the lower bound of Theorem 18. In order to prove the lower bound of Theorem 18, in view of Lemma 13, it suffices to prove, for any $n^* \sim n$ and any constant $\varepsilon > 0$, that in $\mathcal{T}_{n^*,p}$ the number of $T_{a,b}$-extensions of the root is at least $(1 - \varepsilon)\alpha_n$ with probability $\omega((\log n)^{\alpha}/n)$.

Our argument is based on Lemmas 27 and 28 below, which effectively give lower bounds on the probabilities of the discussed strategies. Recall that $\xi_1, \ldots, \xi_n$ are independent variables with distribution $\text{Bin}(n, p)$ and $[n]_p \subseteq [n]$ is a binomial subset, so that the degrees of the children of the root in $\mathcal{T}_{n,p}$ are $(\xi_i : i \in [n]_p)$.

**Lemma 27.** Fix real numbers $x_1, \ldots, x_k > 0$ such that $x := x_1 + \cdots + x_k < 1$. Let $1 \leq pn \ll \log n$. For any constant $\varepsilon > 0$, with probability at least $n^{-x+o(1)}$ there exist distinct $v_1, \ldots, v_k \in [n]_p$ with

$$\xi_{v_1} \geq (x_1 - \varepsilon) D, \ldots, \xi_{v_k} \geq (x_k - \varepsilon) D. \tag{93}$$

**Proof.** We can assume $\varepsilon < \min\{x_1, \ldots, x_k\}$, since decreasing $\varepsilon$ just makes the claim stronger. Cover the interval $[\varepsilon, x]$ with intervals $[j\varepsilon, (j+1)\varepsilon), j \in [J]$, where $J := \lceil \varepsilon^{-1} x \rceil$. For $j \in [J]$ let $c_j = \{i \in [k] : x_i \in [j\varepsilon, (j+1)\varepsilon)\}$. This implies that

$$\sum_{j=1}^{J} c_j \varepsilon \leq \sum_{i=1}^{k} x_i = x. \tag{94}$$
For \( j \in [J] \), let

\[ V_j := \{ v \in [n] : \xi_v \in [j\varepsilon D, (j+1)\varepsilon D) \} \]

and let \( \mathcal{E}_j \) be the event that \([n]_p \cap V_j\] = \( c_j \). Let \( \mathcal{E} := \bigcap_{j=1}^{J} \mathcal{E}_j \). Note that if \( \mathcal{E} \) holds, then there exist \( v_1, \ldots, v_k \in [n]_p \) satisfying (93). It remains to show that \( \mathbb{P}(\mathcal{E}) \geq n^{-x+o(1)} \).

We claim that for every \( j \in [J] \) there is an integer \( w_j = n^{-1-j\varepsilon+o(1)} \) satisfying

\[ |V_j| \geq w_j \quad \text{whp.} \tag{95} \]

Note that \( x < 1 \) implies \( w_j \geq n^{-1-j\varepsilon+o(1)} \geq n^{-x+o(1)} \to \infty \).

To prove (95), note that \( |V_j| \sim \text{Bin}(n, \pi) \), where \( \pi \geq n^{-j\varepsilon+o(1)} \), by inequality (24). Since \( \mathbb{E}|V_j| \geq n^{-1-j\varepsilon+o(1)} \to \infty \), by Chebyshev's inequality, say, we have whp \( |V_j| \geq (\mathbb{E}|V_j|)/2 \geq n^{-1-j\varepsilon+o(1)} \), which proves (95). We further condition on \( |V_j| \geq w_j \) for every \( j \in [J] \) and treat the sets \( V_j \) as deterministic. We have

\[ \mathbb{P}(\mathcal{E}_j) = \frac{|V_j|}{c_j} \Big( p^{c_j}(1-p)^{|V_j| - c_j} \Big)^{w_j} \geq \frac{w_j}{c_j} \Big( p^{c_j}(1-p)^n \Big)^{w_j} = (pn)^{c_j \log n} n^{-j\varepsilon + o(1)} \geq n^{-c_j j^{\varepsilon+o(1)}}. \]

As events \( \mathcal{E}_1, \ldots, \mathcal{E}_J \) are independent, using inequality (94) we conclude

\[ \mathbb{P}(\mathcal{E}) = \prod_{j \in [J]} \mathbb{P}(\mathcal{E}_j) \geq n^{-\sum c_j j^{\varepsilon+o(1)}} \geq n^{-x+o(1)} , \]

which completes the proof of Lemma 27. \( \square \)

**Lemma 28.** Fix \( x_0 \in (0, 1) \) and \( \delta > 0 \). Let \( 1 \ll pn \ll \log n \). With probability at least \( n^{-x_0+o(1)} \) we have

\[ |\{ v \in [n]_p : \xi_v \geq (1-\delta)pn \}| \geq x_0(1-\delta)D(n,p). \]

**Proof.** Writing \( X = |\{ v \in [n]_p : \xi_v \geq (1-\delta)pn \}| \), we have \( X \sim \text{Bin}(n, p') \), where, in view of \( pn \to \infty \), by Chebyshev’s inequality,

\[ p' = p \cdot \mathbb{P}(\xi_1 \geq (1-\delta)pn) \sim p. \]

Recall the definition of \( D(n, p) \) from (7). From \( p' \sim p \) and \( pn \ll \log n \) it easily follows that for sufficiently large \( n \) we have

\[ x_0 D(n, p') \geq x_0(1-\delta)D(n, p). \]

Combining this with inequality (24) we obtain that for sufficiently large \( n \)

\[ \mathbb{P}(X \geq x_0(1-\delta)D(n, p)) \geq \mathbb{P}(X \geq x_0 D(n, p')) \geq n^{-x_0+o(1)} , \tag{96} \]

completing the proof of Lemma 28. \( \square \)

We are ready to prove the lower bound of Theorem 18.

**Proof of the lower bound of Theorem 18.** Let \( n^* = n^*(n) \sim n \) be given by Lemma 13. Write \( D = D(n, p) \) and \( D^* = D(n^*, p) \). Note that \( n^* \sim n \) and \( 1 \ll pn \ll \log n \) imply

\[ pn^* \sim pn \to \infty \quad \text{and} \quad D^* \sim D \to \infty. \tag{97} \]

Note that

\[ f_T(T_{n^*, p}) = \sum_{\text{distinct } v_1, \ldots, v_k \in [n^*_p]} \prod_{j \in [a]} (\xi_{v_j})_b , \]

for
where \([n^*]_p\) is a random subset of \([n^*]\) with every element included independently with probability \(p\) and \(\xi_1, \ldots, \xi_{n^*}\) are independent with distribution \(\text{Bin}(n^*, p)\). Recall that \(D^{b-1}/(pn)^k \to L \in [0, \infty]\). Our goal is to show that, for every fixed \(\epsilon > 0\), we have

\[
\mathbb{P} \left( f_T(T^*_{n^*}, p) \geq (1 - \epsilon)[D(pm)^b]_a \right) \geq \frac{(\log n)^3}{n} \quad \text{if } L = 0,
\]

\[
\mathbb{P} \left( f_T(T^*_{n^*}, p) \geq (1 - \epsilon)D^{ab} \sup f_{a,b}(\Lambda) \right) \geq \frac{(\log n)^3}{n} \quad \text{if } L = \infty,
\]

\[
\mathbb{P} \left( f_T(T^*_{n^*}, p) \geq (1 - \epsilon)[D(pm)^b]_a \sup F_L(\Lambda) \right) \geq \frac{(\log n)^3}{n} \quad \text{if } L \in (0, \infty),
\]

where the function \(F_L\) is defined as in (72) and \(f_{a,b}(\Lambda)\) are image sets of respective functions. Recall that (75) from Proposition 19 gives

\[
[D(pm)^b]_a \sup F_L(\Lambda) \sim \begin{cases} [D(pm)^b]_a, & L \leq 1 \\ D^{ab} \sup f_{a,b}(\Lambda), & L \geq C_{a,b} \end{cases}
\]

Hence, once estimates (98)–(100) are established, the auxiliary ‘transfer result’ Lemma 13 will imply the lower bound in Theorem 18.

To prove (98), we defer the choice of \(\delta = \delta(\varepsilon, a, b) > 0\). Invoking Lemma 28 with \(x_0 = 1 - \delta\), it follows that, with probability at least

\[
(n^*)^{\delta-1+o(1)} = n^{\delta-1+o(1)} \geq \frac{(\log n)^3}{n},
\]

we have \(|\{i \in [n^*]_p : \xi_i \geq (1 - \delta)pm^*\}| \geq (1 - \delta)^2D^*\) and therefore

\[
f_T(T^*_{n^*}, p) \geq ((1 - \delta)^2D^*)_a ((((1 - \delta)pm^*)_b)^a
\]

\[
\sim ((1 - \delta)^2D[(1 - \delta)pm]^b)_a
\]

\[
\geq (1 - \delta)^{ab+2}[D(pm)^b]_a,
\]

which, by choosing \(\delta\) small enough such that \((1 - \delta)^{ab+2} > 1 - \varepsilon\), establishes (98).

To prove (99), we again defer the choice of \(\delta = \delta(\varepsilon, a, b) > 0\), and use that by continuity of \(f_{a,b}\) there is an integer \(k \geq 1\) and a vector \((x_1, \ldots, x_k) \in (0, 1)^k\) such that \(\sum_{i \geq 1} x_i < 1\) and \(f_{a,b}(x_1, \ldots, x_k) \geq (1 - \delta)\sup f_{a,b}(\Lambda)\). Let \(\delta' = \delta \min\{x_1, \ldots, x_k\}\). Lemma 27 implies that, with probability at least

\[
(n^*)^{-\sum_{i \geq 1} x_i + o(1)} = n^{-\sum_{i \geq 1} x_i + o(1)} \geq \frac{(\log n)^3}{n},
\]

there exist distinct \(v_1, \ldots, v_k \in [n^*]_p\) with \(\xi_{v_i} \geq (x_i - \delta')D^* \geq (1 - \delta)x_iD^*\) and therefore

\[
f_T(T^*_{n^*}, p) = \sum_{\text{distinct } i_1, \ldots, i_a \in [k], j \in [a]} \prod_{j \in [a]} ((1 - \delta)x_iD^*)_b
\]

\[
\sim (1 - \delta)^{ab} D^{ab} f_{a,b}(x_1, \ldots, x_k)
\]

\[
\geq (1 - \delta)^{ab+1} D^{ab} \sup f_{a,b}(\Lambda),
\]

which, by choosing \(\delta\) small enough such that \((1 - \delta)^{ab+1} > 1 - \varepsilon\), establishes (99).

Finally we prove the ‘hybrid’ case (100), where we again defer the choice of \(\delta = \delta(\varepsilon, a, b) > 0\). By continuity of \(F_C\), there is an integer \(k \geq 0\) and \(x = (x_0, x_1, \ldots, x_k) \in (0, 1)^{k+1}\) such that \(\sum_{i \geq 0} x_i < 1\) and \(F_C(x) \geq (1 - \delta)\sup F_C(\Lambda)\). Since \(\sum_{i \geq 0} x_i < 1\), Invoking Lemma 27 (with \(\varepsilon = \delta \min\{x_1, \ldots, x_k\}\)) and Lemma 28 then implies that, with probability at least

\[
(n^*)^{-\sum_{i \geq 0} x_i + o(1)} = n^{-\sum_{i \geq 0} x_i + o(1)} \geq \frac{(\log n)^3}{n},
\]
there exist distinct \(v_1, \ldots, v_k \in [n^*]_p\) with \(\xi_{v_i} \geq (1-\delta)x_i D^\ast\) for \(i \in [k]\) and at the same time there are at least \(x_0(1-\delta)D^\ast - k\) elements \(v \in [n^*]_p \setminus \{v_1, \ldots, v_k\}\) such that \(\xi_v \geq (1-\delta)pn^\ast\). This implies, using (97), that \(f_T(T_{n^*})\) is at least

\[
\sum_{m=0}^{a} \binom{a}{m} (x_0(1-\delta)D^\ast - k)a^{-m} \sum_{\text{distinct } i_1, \ldots, i_m \in [k], j \in [m]} ((1-\delta)x_{i_j} D^\ast_b^j) \lesssim \sum_{m=0}^{a} \binom{a}{m} (x_0(1-\delta)D((1-\delta)pn^\ast)b^{a-m} \cdot (1-\delta)^mb^f_m(x_1, \ldots, x_k) D^{mb} \geq (1-\delta)^{a(b+1)} D(pn^\ast)^b a \sum_{m=0}^{a} \binom{a}{m} \left( D^{b-1}(pn^\ast)^b \right)^m a^{-m} f_m(x_1, \ldots, x_k) \lesssim (1-\delta)^{a(b+1)+1} D(pn^\ast)^b \sup F_C(\Lambda). \]

By choosing \(\delta\) small enough such that \((1-\delta)^{a(b+1)+1} > 1 - \varepsilon\), we conclude that (100) holds, which completes the proof of the lower bound of Theorem 18.

7. Minimum tree extension counts

In this paper we always studied the maximum rooted tree extension count, but it is also natural to ask about the minimum count instead. Below we record that are methods provide the following answer to this extreme value theory question, where the ‘sparse’ case \(pn \ll \log n\) is degenerate due to isolated vertices in \(\mathbb{G}_{n,p}\) (since this is not our main focus, we did not investigate the finer details of the phase transition of \(\min_{v \in [n]} X_v\), i.e., how it decreases from order \(\mu_T = \Theta((pn)^\varepsilon T)\) down to zero).

**Theorem 29** (Minimum for general trees, dense case). Fix a rooted tree \(T\), with root degree \(a\). If \(pn \gg \log n\), then

\[
\frac{\mu_T - \min_{v \in [n]} X_v}{\sigma_T \sqrt{2\log n}} \xrightarrow{p} 1, \tag{101}
\]

where the variance \(\sigma_T^2\) is as in Theorem 1. If \(pn \ll \log n\), then

\[
\min_{v \in [n]} X_v \xrightarrow{p} 0. \tag{102}
\]

**Proof.** In the ‘dense’ case \(pn \gg \log n\), the minimum and maximum degree of \(\mathbb{G}_{n,p}\) both deviate by the same amount from asymptotics (25), modulo the sign. Hence (101) follows from (5) in the same way as (4) of Theorem 1 follows from (5) of Proposition 2 (see the proof in Section 3.1).

In the ‘sparse’ case \(pn \ll \log n\) it is well-known that the minimum degree of \(\mathbb{G}_{n,p}\) whp equals zero (see, e.g., [Bo1, Exercise 3.2]), which immediately implies (102). \(\square\)

**References**


Appendix A. Deferred routine proofs

In this appendix we give the deferred proofs of Propositions 7, 9, 10, 11 and Lemma 8 (see Section 2): these are conceptually routine, but we include them for completeness.

Proof of Proposition 7. Recalling $k = (1 + \eta)pn$, by using an approximation by a point probability of the Poisson distribution with mean $pn$ (see [Bol01, eq. (1.14)]) and Stirling’s formula (see [Bol01, eq. (1.4)]) we obtain

$$
\mathbb{P}(X \geq k) \geq \mathbb{P}(X = k) \geq \left(\frac{pm}{k}\right)^k \cdot e^{-(pm)^2 + k^2} = \frac{1}{\sqrt{2\pi k}} \left(\frac{e^{pm}}{k}\right)^k \cdot e^{-pm + o(1)}
$$

$$
= \exp \left(-k \log(k/(pm)) + k - pn - \log \sqrt{2\pi k} + o(1)\right)
$$

$$
= \exp \left(-pm\phi(\eta) + O(\log k)\right),
$$
which establishes inequality (22).

**Proof of Lemma 8.** We start with inequality (23). Using the simplified Chernoff bound (21) and $\frac{\log n}{pn} \to \infty$, we obtain

$$-\log \mathbb{P}(\xi \geq \alpha D) \geq \frac{\alpha \log n}{\log \frac{\log n}{pn}} \cdot \left( \log \frac{\log n}{pn} - \log \log \frac{\log n}{pn} + \log \alpha - 1 \right)$$

$$\sim \alpha \log n,$$

which establishes inequality (23).

Next we prove inequality (24). Note that condition $1 \leq pn \ll \log n$ implies $\alpha \frac{\log n}{pn} \gg 1$ and $1 \ll D \ll \log n$.

By Proposition 7 and the asymptotics (28), it thus follows that

$$-\log \mathbb{P}(\xi \geq \alpha D) \leq -\log \mathbb{P}(\xi = \lceil \alpha D \rceil)$$

$$\leq pn \phi \left( \frac{\lceil \alpha D \rceil}{pn} - 1 \right) + O(\log \lceil \alpha D \rceil)$$

$$\sim \alpha D \log \frac{\alpha D}{pn}$$

$$= \frac{\alpha \log n}{\log \frac{\log n}{pn}} \cdot \left( \log \frac{\log n}{pn} - \log \log \frac{\log n}{pn} + \log \alpha \right)$$

$$\sim \alpha \log n,$$

establishing that $\mathbb{P}(\xi \geq \alpha D) \geq n^{-\alpha + o(1)}$. Combining this lower bound with the upper bound (23), it readily follows (using that $\alpha, \varepsilon > 0$ are constants) that

$$\mathbb{P}(\alpha D \leq \xi < (\alpha + \varepsilon) D) \geq n^{-\alpha + o(1)} - n^{-\alpha + \varepsilon + o(1)} = n^{-\alpha + o(1)},$$

completing the proof of inequality (24). □

The proofs of Propositions 9 and 10 rely on the following maximum degree criterion, which follows from Theorem 3.2 and Theorem 3.1 parts (i) and (ii) in [Bol01].

**Theorem 30 ([Bol01]).** Assume that $pqn \geq 1$. For any natural number $k = k(n)$,

$$\mathbb{P}(\Delta(G_{n,p}) \geq k) \to \begin{cases} 0 & \text{if } n \cdot \mathbb{P}(\text{Bin}(n - 1, p) \geq k) \to 0, \\ 1 & \text{if } n \cdot \mathbb{P}(\text{Bin}(n - 1, p) \geq k) \to \infty. \end{cases}$$

**Proof of Proposition 9.** It is enough to prove (25) for the maximum degree $\Delta$, since the other limit follows by considering random variable $(n - 1) - \delta$, which happens to be the maximum degree in the complement of $G_{n,p}$ which is distributed as $G_{n,q}$ (note that both the condition $pqn \gg \log n$ and the denominator $\sqrt{2pqn \log n}$ are unchanged if we swap the roles of $p$ and $q$).

Under the additional assumption $pqn \gg \log^3 n$, the maximum degree limit in (25) follows from [Bol01, Corollary 3.4]. So we can henceforth assume $\log n \ll pqn = O(\log^3 n)$.

To complete the proof of Proposition 9 it remains to show, for an arbitrary small number $\varepsilon > 0$, that

$$pn + h_\varepsilon \leq \Delta \leq pn + h_+ \quad \text{whp},$$

(103)
where $h_+ := \sqrt{(2 \pm \varepsilon)pqn \log n}$. For this we apply Theorem 30. For $X \sim \text{Bin}(n-1, p)$ it thus suffices to check that the following two inequalities hold:

$$\Pr(X \geq pn + h_+) \ll 1/n,$$

$$\Pr(X \geq pn + h_-) \gg 1/n.$$  \hspace{1cm} (104)

To prove (104), note that the variance $\sigma^2$ of $X$ satisfies $\sigma^2 \sim pqn \gg \log n$ and therefore $h_+ \sim \sigma \sqrt{(2 + \varepsilon) \log n} \ll \sigma^2$. A standard Bernstein bound (see [JLR00, eq. (2.14)]) gives

$$\Pr(X - p(n - 1) \geq h_+) \leq \exp \left( -\frac{\beta + 1}{2(\sigma^2 + h_+/3)} \right) = e^{-(1+\varepsilon/2+o(1)) \log n} \ll \frac{1}{n},$$

which implies (104).

To prove (105), we apply the following inequality (see [Bol01, Theorem 1.5]), which we simplify for ease of application (formally weakening it using the inequalities $p, q \leq 1$). Namely, for any integer $k = pn + h < n$ such that $h > 0$ and $pn \geq 1$, we have

$$\Pr(\text{Bin}(n, p) = k) \geq \frac{1}{\sqrt{2\pi pqn}} \exp \left( -\frac{h^2}{2pqn} \left( 1 + \frac{1}{h} - \frac{2h}{3pqn} - \frac{1}{h} \right) \right),$$  \hspace{1cm} (106)

where $\beta := 1/(12k) + 1/(12(n-k))$. We apply inequality (106) with $k := [pn + h_-]$. Since $nq \geq pqn \gg h_-$, it is easy to check that $n - k \to \infty$ and therefore $\beta \to 0$. Moreover $h = k - pn = h_- + O(1) \sim \sqrt{(2 - \varepsilon)pqn \log n}$ whence $1 \ll h \ll pqn$. Now (106) implies

$$\Pr(X \geq pn + h_-) \geq \Pr(\text{Bin}(n, p) = k) \geq \frac{1}{\sqrt{2\pi pqn}} \exp \left( -\frac{(2 - \varepsilon) \log n}{2} (1 + o(1)) \right).$$

Recalling that we also assume $pqn = O(\log^3 n)$, we therefore obtain

$$\Pr(X \geq pn + h_-) \geq \exp \left( -[1 - \varepsilon / 2 + o(1)] \log n + O(\log \log n) \right) \gg 1/n,$$

which implies (105), completing the proof of estimate (103) and thus of Proposition 9. \hfill \Box

**Proof of Proposition 10.** First, the asymptotics (27) follow directly from (29). Turning to the remaining proof of (26), set $\varepsilon := 1/(\log \log n)$ and let

$$k_\pm := (1 + \eta_\pm)pn, \quad \text{with} \quad \eta_\pm := \phi^{-1}\left(1 \pm \varepsilon \frac{\log n}{pn}\right).$$

Using asymptotics (29) one can check that $1 \ll k_- = O(\log n)$, and therefore

$$1 \ll \log k_- = O(\log \log n).$$

By monotonicity of $\phi^{-1}$ and assumption $1 \ll pn = O(\log n)$ we have $\eta_- = \Omega(1)$ which implies the following two sets of inequalities:

$$\alpha_n \geq \eta_-pn = \Omega(pn) \to \infty,$$

$$pn \leq [k_-] \leq k_-.$$  \hspace{1cm} (107)

(108)

Hence Proposition 7 implies that

$$\log \Pr(\text{Bin}(n, p) \geq [k_-]) \geq -pn\phi\left(\frac{[k_-]}{pn} - 1\right) + O(\log k_-)$$

from which it readily follows that

$$n \cdot \Pr(\text{Bin}(n-1, p) \geq [k_-] - 1) \geq n \cdot \Pr(\text{Bin}(n, p) \geq [k_-]) \to \infty.$$
On the other hand, the Chernoff bound \((20)\) implies
\[
n \cdot \mathbb{P}(\text{Bin}(n-1, p) \geq k_+) \leq ne^{-p(n-1)\phi(\epsilon)} = \exp \left( \log n - (1 + \epsilon) \frac{p(n-1) \log n}{pn} \right) \to 0.
\]
Consequently, by Theorem 30 it follows that whp \([k_-] - 1 \leq \Delta < k_+\). Note that
\[
k_+ - pn = pn\phi^{-1} \left( (1 + \epsilon) \frac{\log n}{p(n-1)} \right) \sim pn\phi^{-1} \left( \frac{\log n}{pn} \right) = \alpha_n.
\]
where \(\sim\) can be justified using the subsequence principle (which allows us to assume that both arguments of \(\phi^{-1}\) converge to \(c \in [0, \infty]\); then we use continuity of function \(\phi^{-1}\) when \(c < \infty\), and the asymptotics \((29)\) of \(\phi^{-1}\) when \(c = \infty\)). In view of \((107)\) it also follows that \([k_-] - 1 - pn \sim \alpha_n\), completing the proof of \((26)\).

\[\square\]

**Proof of Proposition 11.** Writing \(N := (n-1)_{v_T - 1}\), let \(T_1, \ldots, T_N\) denote all \(T\)-extensions of vertex \(v\) in \(K_n\). For convenience we will sometimes treat extensions as subgraphs of \(K_n\) (remembering that several extensions correspond to the same subgraph). Let \(I_i\) denote the indicator random variable for the event that \(T_i \subseteq \mathcal{G}_{n,p}\). Since \(\mathbb{E}I_i = p^{e_T}\) and \(X_v = I_1 + \cdots + I_N\), using \(e_T = v_T - 1\) it follows that the expectation of \(X_v\) satisfies
\[
\mu_T = \mathbb{E}X_v = Np^{e_T} = (n-1)e_Tp^{e_T} = (pn)^{e_T} (1 + O(1/n)).
\]

We now turn to the variance \(\sigma_T^2 = \text{Var } X_v\). Writing \(\text{Cov}(I_i, I_j) = \mathbb{E}(I_i - p^{e_T})(I_j - p^{e_T})\), by observing that \(\text{Cov}(I_i, I_j) = 0\) whenever \(e_{T_i \cap T_j} = 0\) it follows that
\[
\sigma_T^2 = \mathbb{E} \left( \sum_i (I_i - p^{e_T}) \right)^2 = \sum_{i,j} \text{Cov}(I_i, I_j) = \sum_{i,j:e_{T_i \cap T_j} > 0} \text{Cov}(I_i, I_j). \tag{109}
\]

We will see that the leading term in \((109)\) comes from the pairs \((i, j)\) for which \(T_i\) and \(T_j\) overlap in exactly one edge that is incident to the root \(v\). The number of such pairs is \(a^n n^{2e_T - 1}(1 + o(1))\), since, having chosen \(T_i\) (in one of \(N \sim n^{e_T}\) ways), there are \(a\) choices of the overlapping edge \(e\) in \(T_i\) and \(a\) choices of which edge of \(T\) is mapped to \(e\) in \(T_j\), prescribing exactly one vertex in \(T_j\) and leaving \(n^{e_T - 1}(1 + o(1))\) choices for the remaining vertices of \(T_j\). Given \(k = 2, \ldots, e_T\), the number of pairs \((i, j)\) with \(e_{T_i \cap T_j} = k\) is \(O(n^{2e_T - k})\), since fixing \(k\) edges in \(T_i\), due to the tree structure fixes at least \(k\) vertices of \(T_j\). Moreover, the number of pairs \((i, j)\) such that \(T_i\) and \(T_j\) share exactly one edge, but this edge is not incident to the root \(v\), is \(O(n^{2e_T - 2})\). Since for any pair sharing \(k\) edges we have
\[
\text{Cov}(I_i, I_j) = p^{2e_T - k}(1 - p^k) \asymp p^{2e_T - k} q,
\]
it follows that the variance of \(X_v\) satisfies
\[
\sigma_T^2 = a^n n^{2e_T - 1} p^{2e_T - 1} q (1 + o(1)) + O \left( n^{2e_T - 2} p^{2e_T - 1} q \right) + O \left( \sum_{k=2}^{e_T} (pn)^{2e_T - k} q \right)
\]
\[
= a^n n^{2e_T - 1} p^{2e_T - 1} q \left( 1 + O \left( \frac{1}{pn} \right) \right) \sim \frac{a^2 \mu_T^2 q}{pn},
\]
completing the proof of Proposition 11. \[\square\]
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