The lower tail: Poisson approximation revisited

Lutz Warnke
University of Cambridge

Joint work with Svante Janson

Motivating Example

Important quantity in random graph theory

- X_H = number of copies of H in $G_{n,p}$
- H is a fixed graph (triangle, 4-cycle, r-clique, etc)

Classical result (Janson–Łuczak–Ruciński, 1987)

Let $\Phi_H = \min_{J \subseteq H} \mathbb{E} X_J$. If $n \geq n_0(H)$, then

$$\mathbb{P}(X_H = 0) = \exp\bigl(-\Theta(\Phi_H)\bigr)$$

This talk: lower tail problem (Janson–W., 2014+)

If $\varepsilon^2 \Phi_H \geq c_0(H)$ and $n \geq n_0(H)$, then

$$\mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E} X_H) = \exp\bigl(-\Theta(\varepsilon^2 \Phi_H)\bigr)$$

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Why should we care?

- Test/develop tools in combinatorial probability (for tail behaviour)
- Interesting question: concentration of measure + large deviations

Talk-Outline

1. **State Janson's Inequality**

   $\mathbb{P}(X \leq (1-\varepsilon)\mathbb{E}X) \leq \exp\big(-\phi(-\varepsilon)\Psi_X\big)$

2. **Matching lower-bound in Poisson case**

   $\mathbb{P}(X \leq (1-\varepsilon)\mathbb{E}X) \geq \exp\big(-(1+o(1))\phi(-\varepsilon)\Psi_X\big)$

3. **Lower-bound reduction to Poisson case (for subgraphs)**

   $\mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E}X_H) \geq \Omega(1) \cdot \mathbb{P}(X_J \leq (1-\varepsilon)\mathbb{E}X_J)$

4. **Open problem for triangle counts in $G_{n,p}$**

   $-\log \mathbb{P}(X_{K_3} = 0) \sim f(c)n^{3/2}$ for $p = cn^{-1/2}$?

Janson's inequality

Binomial random subsets framework

- Γ_p = random subset: each $i \in \Gamma$ included indep. with probability p
- $X = \sum_{A \in \mathcal{S}} I_A$, where $I_A = \mathbf{1}_{\{A \subseteq \Gamma_p\}}$
- Parameter δ measures how dependent the I_A are ($\delta = 0$ if independent)

- Special structure: X is sum of *increasing* indicators
- $X =$ "Number elements from $\mathcal{S}$ which are contained in Γ_p"

Janson's inequality (Janson, 1989)

Let $\phi(x) = (1+x)\log(1+x) - x$ and $\mu = \mathbb{E}X$. Then

$$\mathbb{P}(X \leq (1-\varepsilon)\mu) \leq \exp\big(-\phi(-\varepsilon)\mu/(1+\delta)\big)$$

- Widely used in combinatorial probability/random graph theory
- Reduces to Chernoff/Bernstein bounds in case of $\delta = 0$

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- For special case of independent summands ($\delta = 0$) best possible (as Chernoff bounds are sharp)

Goal of this talk

Prove that Janson's inequality is very often (close to) best possible

- In the 'weekly dependent' case $\delta = o(1)$ we, e.g., want to show

$$\mathbb{P}(X \le (1-\varepsilon)\mu) \ge \exp\bigl(-(1+o(1))\phi(-\varepsilon)\mu\bigr)$$

Main technical result

Janson's inequality sharp if X approx. Poisson (Janson–W., 2014+)

Let $\mu = \mathbb{E}X$, $\pi = \max_{A \in \mathcal{S}} \mathbb{E} I_A$ and $\phi(x) = (1+x)\log(1+x) - x$.
If $\max\{\delta, \pi\} \to 0$ and $\varepsilon^2\mu \to \infty$, then

$$-\log \mathbb{P}(X \le (1-\varepsilon)\mu) \sim \phi(-\varepsilon)\mu = \Theta(\varepsilon^2\mu)$$

Remarks

- Condition $\varepsilon^2\mu \to \infty$ is natural: study exponentially small probabilities
- Condition $\max\{\delta, \pi\} \to 0$ is natural: implies $d_{\mathrm{TV}}(X, \mathrm{Po}(\mu)) \to 0$
- Stronger than usual: ε is *not* fixed
- When $\delta = O(1)$: determine exponent up to constant factors

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Proof remarks

- Our contribution: 'matching lower bound'
- Special case $\varepsilon = 1$ has simple FKG-based proof (JŁR, 1987)
- We use Hoelder's inequality, Laplace transform, correlation ineq. etc

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If $\max\{\delta, \pi\} \to 0$ and $\varepsilon^2 \mu \to \infty$, then

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What about $\delta \to \infty$ case?

- For subgraph counts we can always reduce to (weakly) Poisson case
- There is $J \subseteq H$ with $\delta(J) = O(1)$, so that

$$\mathbb{P}(X_H \le (1-\varepsilon)\mathbb{E}X_H) \ge \Omega(1) \cdot \mathbb{P}(X_J \le (1-\varepsilon)\mathbb{E}X_J)$$

Case-study: 'house of santa claus' graph

$H =$ 'house of santa claus'

So far we know

$$-\log \mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E}X_H) \sim \begin{cases} \phi(-\varepsilon)\mathbb{E}X_H, & \text{if } p \ll n^{-1/2}, \\ \Theta(\varepsilon^2 \mathbb{E}X_H), & \text{if } p = O(n^{-1/2}). \end{cases}$$

- For $p \gg n^{-1/2}$ our discussed methods break: $\delta(H) \to \infty$
- Fact: for $n^{-1/2} \leq p \leq n^{-2/5}$ Janson's inequality gives

$$\mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E}X_H) \leq \exp(-c\varepsilon^2 \mathbb{E}X_{K_4})$$

Tantalizing observation for $n^{-1/2} \leq p \leq n^{-2/5}$

K_4 is in weakly Poisson case: $\delta(K_4) = O(1)$, so

$$\mathbb{P}(X_{K_4} \leq (1-\varepsilon)\mathbb{E}X_{K_4}) \geq \exp(-C\varepsilon^2 \mathbb{E}X_{K_4})$$

Bootstrapping lower bounds: $\delta \to \infty$ case

Reduction to $\delta = O(1)$ case

- Writing $\mathcal{D}_J =$ "$X_J \le (1-\varepsilon)\mathbb{E}X_J$", we aim at

$$\mathbb{P}(\mathcal{D}_H) \ge \underbrace{\mathbb{P}(\mathcal{D}_{K_4})}_{\substack{\text{apply lower bound} \\ \text{using } \delta(K_4) = O(1)}} \cdot \underbrace{\mathbb{P}(\mathcal{D}_H \mid \mathcal{D}_{K_4})}_{\text{hope that } \Omega(1)} \ge \exp\big(-\Theta(\varepsilon^2 \mathbb{E}X_{K_4})\big)$$

- Idea: "conditioning on $\mathcal{D}_{K_4}$ converts rare event $\mathcal{D}_H$ into typical one"

Intuition for $H=$ 'house of santa claus'

- 'Too few' K_4-copies *typically* implies 'too few' H-copies (*)
- $\mathbb{E}(X_H \mid \mathcal{D}_{K_4}) \le (1-\varepsilon)\mathbb{E}X_H$

- !!! We only managed to prove weaker variants of (*) !!!
- Calculating *conditional* second moment seems difficult

Lower tail for subgraph counts (Janson–W., 2014+)

Let $\Phi_H = \min_{J \subseteq H} \mathbb{E} X_J$. If $\varepsilon^2 \Phi_H \geq c_0(H)$ and $n \geq n_0(H)$, then

$$\mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E}X_H) = \exp\big(-\Theta(\varepsilon^2 \Phi_H)\big)$$

Bootstrapping approach can always be applied

- Enough to focus on the subgraph $J \subseteq H$ with $\Phi_H = \mathbb{E}X_J$

Rate of decay consistent with normal approximation:

- $\varepsilon^2 \Phi_H = \Theta((\varepsilon \mathbb{E} X_H)^2 / \operatorname{Var} X_H)$

Subgraph Counts 2/2

With more care we can, e.g., also establish the following result

Gaussian behavior for 2-balanced graphs (Janson–W., 2014+)

Assume that H is "2-balanced" (a tree, cycle, clique, hypercube, etc)
If $(\varepsilon \mathbb{E} X_H)^2 \gg \operatorname{Var} X_H$ and $\varepsilon \ll 1$, then

$$-\log \mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E} X_H) \sim \frac{(\varepsilon \mathbb{E} X_H)^2}{2 \operatorname{Var} X_H},$$

excluding only the ranges $p = \Theta(n^{-1/m_2(H)})$ and $p = \Theta(1)$.

Summary

Informal summary (Janson–W.)

Janson's inequality is often close to *best possible*

- Large deviation rate function in Poisson case:

$$-\log \mathbb{P}(X \leq (1-\varepsilon)\mathbb{E}X) \sim \varphi(\varepsilon)\mathbb{E}X$$

- Subgraphs example (reduction to Poisson case for lower bound):

$$\mathbb{P}(X_H \leq (1-\varepsilon)\mathbb{E}X_H) = \exp\Big(-\Theta(\varepsilon^2 \min_{J\subseteq G} \mathbb{E}X_J)\Big)$$

Open problem: Triangle counts in $G_{n,p}$ for $p = cn^{-1/2}$

Would be nice to prove $-\log \mathbb{P}(X_{K_3} = 0) \sim f(c)n^{3/2}$

- Know asymptotics of $-\log \mathbb{P}(X_{K_3} = 0)$ for *all* other ranges of p
- Maybe (some variant of) the interpolation method works?