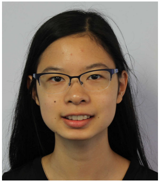
Two-Point Concentration of the Domination Number of Random Graphs

Lutz Warnke

UC San Diego

Joint work with *Tom Bohman* (CMU) and *Emily Zhu* (UCSD)



Topic: Two-Point Concentration

Fundamental Problem

Given a graph-parameter X, which probabilities $p = p(n)$ have the property that $X(G_{n,p})$ is concentrated on two values in the random graph $G_{n,p}$?

- $\mathbb{P}(X(G_{n,p}) \in \{r, r+1\}) \to 1$ for some deterministic $r = r(n,p)$
- Examples: Chromatic Number, Clique + Independence Number

Today

Two-point concentration of *Domination Number* $\gamma(G_{n,p})$

- $\gamma(G_{n,p})$ = size of smallest vertex set K such that in $G_{n,p}$ every vertex $v \notin K$ has at least one neighbor in K
- Fundamental parameter (third example in "Probabilistic Method")
- Two-point concentration studied since 1981

History: Domination Number

Two-Point Concentration of $\gamma(G_{n,p})$

$p = 1/2$	Weber	1981
$p \gg \sqrt{\frac{\log\log n}{\log n}}$	Godbole-Wieland	2001
$p \geq n^{-1/2}(\log n)^2$	Glebov-Liebenau-Szabó	2015

- Range of $p = p(n)$ was believed to be essentially best possible

Conjecture (Glebov-Liebenau-Szabó)

Two-point concentration of $\gamma(G_{n,p})$ *fails* for $p \ll n^{-1/2}$

History: Domination Number

Two-Point Concentration of $\gamma(G_{n,p})$

$p = 1/2$	Weber	1981
$p \gg \sqrt{\frac{\log\log n}{\log n}}$	Godbole-Wieland	2001
$p \geq n^{-1/2}(\log n)^2$	Glebov-Liebenau-Szabó	2015

- Range of $p = p(n)$ was believed to be essentially best possible

Conjecture (Glebov-Liebenau-Szabó)

Two-point concentration of $\gamma(G_{n,p})$ *fails* for $p \ll n^{-1/2}$

This talk

We disprove the conjecture (it fails around $p = n^{-2/3}$)

Main Results: Domination Number

Two-Point Concentration for $p \geq n^{-2/3+\varepsilon}$ (Bohman-Warnke-Zhu)

If $p \geq n^{-2/3}(\log n)^3$, then there is $r = r(n, p)$ such that
$\mathbb{P}(\gamma(G_{n,p}) \in \{r, r+1\}) \to 1$ as $n \to \infty$

- *Disproves conjecture of Glebov-Liebenau-Szabó*
- Proof: first + second moment method
- *Major new technical obstacle arises for $p \leq n^{-1/2}$:*
  $\longrightarrow$ we overcome by **adapting Janson's Inequality**

Main Results: Domination Number

Two-Point Concentration for $p \geq n^{-2/3+\varepsilon}$ (Bohman-Warnke-Zhu)

If $p \geq n^{-2/3}(\log n)^3$, then there is $r = r(n,p)$ such that $\mathbb{P}(\gamma(G_{n,p}) \in \{r, r+1\}) \to 1$ as $n \to \infty$

- *Disproves conjecture of Glebov-Liebenau-Szabó*
- Proof: first + second moment method
- *Major new technical obstacle arises for $p \leq n^{-1/2}$:*
  $\longrightarrow$ we overcome by **adapting Janson's Inequality**

No Two-Point Concentration for $p \leq n^{-2/3}$ (Bohman-Warnke-Zhu)

If $p \leq n^{-2/3}(\log n)^{2/3}$, then there is $q \in [p, 2p]$ such that $\max_{r \geq 0} \mathbb{P}(\gamma(G_{n,q}) \in \{r, r+1\}) \leq 3/4$ for infinitely many n

- Proof: coupling + discrete derivative argument

Glimpse of Proof 1/3: Second Moment Method

- **Setup (to show existence of dominating sets)**
  - $X = \#$ dominating sets in $G_{n,p}$ of size r
  - Show $\mathbb{E}X \to \infty$ and $\operatorname{Var} X \ll (\mathbb{E}X)^2$ for suitable r
- **Variance calculation**
  - For most sets A, B we essentially need to show

$$\mathbb{P}(\text{A,B both dominate}) \leq (1+o(1))\mathbb{P}(A \text{ dominates})\mathbb{P}(B \text{ dominates})$$

  - Ignoring some details, this reduces to

$$\mathbb{P}(\text{A,B dom. each other}) \leq (1+o(1))\mathbb{P}(A \text{ dom. } B)\mathbb{P}(B \text{ dom. } A)$$

  - Requires **Poisson-Approximation** when A and B are disjoint:

$$\mathbb{P}(X=0) \leq (1+o(1))\exp(-\mathbb{E}X)$$

where $X = \#$ of isolated vertices in random bipartite graph $G_{n,p}[A, B]$

Glimpse of Proof 2/3: Poisson Approximation

- **Goal: Poisson-Approximation**
  - For $X = \#$ of isolated vertices in random bipartite graph $G_{n,p}[A,B]$, want

$$\mathbb{P}(X=0) \le (1+o(1))\exp(-\mathbb{E}X)$$

  - Major Difficulty: many mild dependencies (every $v \in A$ with all $w \in B$)
- **Remarks**
  - This holds 'for free' when $p \gg n^{-1/2}(\log n)$, as then $\mathbb{E}X \to 0$
  - *Standard tools fail for* $p \ll n^{-1/2}$ (where $\mathbb{E}X = n^{1/3+o(1)}$ possible)
    - ★ Method of moments, Stein–Chen method and inclusion-exclusion: work when $\mathbb{E}X$ to not too large
    - ★ Janson's inequality: dependency parameter Δ too large
- **Our Approach**
  - We adapt the general proof of Janson's inequality to the specific situation

Glimpse of Proof 3/3: Adapting Janson's Inequality

- **Goal: Poisson-Approximation**
  - For $X = \#$ of isolated vertices in random bipartite graph $G_{n,p}[A,B]$, want

$$\mathbb{P}(X=0) \le (1+o(1))\exp\big(-\mathbb{E}X\big)$$

  - Major Difficulty: many mild dependencies (every $v \in A$ with all $w \in B$)

- **Our Approach: adapt the proof of Janson's inequality**
  - Taking the mild dependencies into account, we can *improve*

$$\mathbb{P}(X=0) \le \exp\big(-\mathbb{E}X + \Delta/2\big)$$

    to the better bound

$$\mathbb{P}(X=0) \le \exp\big(-\mathbb{E}X + \Delta p \cdot \log(1+\cdots)\big)$$

  - Natural improvement: since $\operatorname{Var} X \le \mathbb{E}X + \Delta \cdot p/(1-p)$
  - Essentially best possible: $\Delta p \cdot \log(1+\cdots) = o(1)$ for $p \ge n^{-2/3}(\log n)^{5/2}$

Summary

Two-point concentration of Domination Number $\gamma(G_{n,p})$

- True for $p \gg n^{-2/3}(\log n)^3$
- Fails for $p \ll n^{-2/3}(\log n)^{2/3}$

- Disproved conjecture of Glebov-Liebenau-Szabó
- Main Proof Ingredient: *adapting Janson's inequality to situation*

Open Problem

- Prove better anti-concentration results for $p \leq n^{-2/3}$

- One approach would be to bound $\max_k \mathbb{P}(\gamma(G_{n,p})) = k)$