Concentration of the chromatic number of sparse random graphs

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Context and Main Question

- **Random graph** $G_{n,p}$: $n$-vertex graph where each of $\binom{n}{2}$ possible edges included independently with probability $p$
- **Chromatic number** $\chi(G)$: minimum number of colors needed to color vertices of $G$ s.t. no two adjacent vertices have same color

**Main Question**
Suppose there is an interval of length $\ell(n, p)$ that contains chromatic number $\chi(G_{n,p})$ with high probability. How small can $\ell(n, p)$ be?
Past Results: constant $p$

**Bollobas 1988**

For constant edge-probability $p \in (0, 1)$, whp

$$\chi(G_{n,p}) = (1 + o(1)) \frac{n}{2 \log_{1/(1-p)} n}.$$ 

So $\ell(n, p) = o(n/ \log_b n)$.

- **Lower bound**: Show largest ISET is of size $(2 + o(1)) \log_{1/(1-p)} n$.
- **Upper bound**: Repeatedly pull out ISET of size $2 \log_{1/(1-p)} n$ until $O(\sqrt{n}/ \log n)$ vertices are left (via Janson’s inequality).
Past Results: all $p$

### Shamir and Spencer 1987

- $\ell(n, p) \leq \omega \sqrt{n}$ for any $p(n)$,
- $\ell(n, p) \leq \omega \sqrt{np \log n}$ for $p = n^{-\alpha}, \alpha \in (0, 1/2)$

**Basic-Idea:** Via Martingale argument to show that whp there exists $\Lambda \geq 0, Z \subseteq V, |Z| \leq \omega \sqrt{n}$

$$\Lambda \leq \chi(G_{n,p}) \leq \Lambda + \chi(G_{n,p}[Z])$$

**Key-Task:** argue that $G_{n,p}[Z]$ is sparse

- For $p = n^{-\alpha}$, easy to remove extra log $n$ term with modern argument
Past Results

log improvement by Alon (and later independently by Scott)

For $p \in [0, 1]$ constant, $\ell(n, p) \leq \omega \sqrt{n} / \log n$

- **Idea**: Repeatedly remove ISET of size $\Theta(\log n)$ from $G_{n,p}[Z]$
- If we use Janson's inequality to pull out the ISET, this only works until $p = n^{-\alpha}$ for some small $\alpha > 0$

Alon and Krivelevich

Let $\epsilon > 0$. If $p \leq n^{-1/2-\epsilon}$, then $\ell(n, p) \leq 2$
New result: Sparse case $p = o(1)$

Surya and Warnke (2022+)

Let $\epsilon > 0$. If $p \geq n^{-1/2 + \epsilon}$, then

$$\ell(n, p) \leq \frac{\omega \sqrt{np}}{\log n}$$

- Use *density argument* instead of large deviation inequalities
More detailed statement: Sparse case $p = o(1)$

Surya and Warnke (2022+)

If $ω\sqrt{np} \gg \log n$, then

$$\ell(n, p) = O\left(\frac{ω\sqrt{np}}{\log(ω\sqrt{np}/\log n)}\right)$$

If $ω\sqrt{np} \ll \log n$, then

$$\ell(n, p) = O\left(\frac{\log n}{\log(\log n/(ω\sqrt{np}))}\right)$$

If $p = n^{-α}, α ∈ (0, 1/2)$ we have $\ell(n, p) = O\left(\frac{ω\sqrt{np}}{\log n}\right)$, extending log improvement of Alon.

Match the best known upper bound up to some constant factor when $p$ constant and $p ≤ n^{-1/2−ε}$
Key Ingredient: Greedy Algorithm

Will focus on controlling $\chi(G_{n,p}[Z])$.

**We use greedy algorithm in two ways**, *exploiting small degree vertices*:

- Pull out largest independent sets until $O\left(\frac{\log n}{p}\right)$ vertices are left, which will have typical size $\approx O\left(\log(\omega \sqrt{np}/\log n)/p\right)$.
  
  ▶ *Refined analysis*: as fewer vertices remain, the independent sets get smaller (exploit that few vertices remain).

- Pick the minimum degree vertex among the remaining vertices, which will have degree $O(\log n)$.

**Chernoff bound + Union bound**: small degree conditions holds whp
Greedy Lemmas

To iteratively pull out largest independent set (until few vertices remain):

**Large independent sets: greedy bound**

Given graph $G$ and $0 < d < 1 < u$ with $\delta(G[S]) \leq d(|S| - 1)$ for all $S \subseteq V(G)$ of size $|S| \geq u$. Then

$$\alpha(G[W]) \geq -\log_{(1-d)(1-1/u)}\left(\frac{|W|}{u}\right)$$

for any $W \subseteq V(G)$ of size $|W| \geq u$.

To color the remaining $O(\log n/p)$ vertices:

**Chromatic number: greedy bound**

Given a graph $G$ with $\delta(G[S]) \leq r$ for all $S \subseteq V(G)$. Then

$$\chi(G) \leq r + 1$$
Large independent sets: greedy bound

Given graph $G$ and $0 < d < 1 < u$ with $\delta(G[S]) \leq d(|S| - 1)$ for all $S \subseteq V(G)$ of size $|S| \geq u$. Then $\alpha(G[W]) \geq -\log(1-d)(1-1/u)(|W|/u)$ for any $W \subseteq V(G)$ of size $|W| \geq u$.

Construct independent set greedily: set $W_0 = W$ and, for $i \geq 1$, pick $w_i \in W_{i-1}$ with minimal degree in $G[W_{i-1}]$ and set

$$W_i = \{v \in W_{i-1} : v \text{ not adjacent to } w_i\}.$$  

If $|W_{i-1}| \geq u$ holds, then $\deg_G(W_i)(w_i) \leq d(|W_i| - 1)$, implying that

$$|W_i| \geq (1 - d)(|W_{i-1}| - 1) \geq (1 - d)(1 - 1/u)|W_{i-1}|.$$
Large independent sets: greedy bound

Given graph $G$ and $0 < d < 1 < u$ with $\delta(G[S]) \leq d(|S| - 1)$ for all $S \subseteq V(G)$ of size $|S| \geq u$. Then $\alpha(G[W]) \geq -\log_{(1-d)(1-1/u)} (|W|/u)$ for any $W \subseteq V(G)$ of size $|W| \geq u$.

Construct independent set greedily: set $W_0 = W$ and, for $i \geq 1$, pick $w_i \in W_{i-1}$ with minimal degree in $G[W_{i-1}]$ and set

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If $|W_{i-1}| \geq u$ holds, then $\deg_{G[W_i]}(w_i) \leq d(|W_i| - 1)$, implying that

$$|W_i| \geq (1 - d)(|W_{i-1}| - 1) \geq (1 - d)(1 - 1/u)|W_{i-1}|.$$

So $W_i$ is non-empty for

$$i - 1 \leq -\log_{(1-d)(1-1/u)} (|W|/u) =: I(|W|),$$

so we terminate with an independent set $\{w_1, \ldots, w_j\} \subseteq W$ of size $j \geq \lceil I(|W|) + 1 \rceil \geq I(|W|)$. 
Very dense case 1 \(- p = n^{-\Omega(1)}\)

- **Heuristic**: Optimal colouring is obtained by taking as many disjoint \(\alpha\)-ISETs as possible, then covering the rest with \((\alpha - 1)\)-ISETs.

- **Main source of fluctuation**: number of \(\alpha\)-ISETs

**Conjecture**

\[
\left(\log n\right)^{1/(r)} n^{-2/r} \ll 1 - p \ll n^{-2/(r+1)} \text{ for some integer } r \geq 1. \]

Let \(\mu_{r+1} = \mu_{r+1}(n, p) := \binom{n}{r+1}(1 - p)^{r+1} \binom{r+1}{2} \) be the expected number of \(r + 1\)-ISET. Then

\[
\ell(n, p) = \omega \sqrt{\mu_{r+1}}
\]
Very dense case $1 - p = n^{-\Omega(1)}$ conjecture

Figure: Conjecture predicts if $n^2(1 - p) = n^{x + o(1)}$, then $\ell(n, p) = n^{y + o(1)}$
Concentration result: \(1 - p = O(1/n)\)

- **Number of ISET of size \(\geq 3\) is negligible:**
  Problem reduces to studying maximum matching on complement

- **Source of fluctuation in maximum matching on \(G_{n,q}\):**
  *Fluctuation of isolated edges*

**Theorem Surya and Warnke (2022+)**

\[ Cn\sqrt{q} \leq \ell(n, p) \leq \omega n\sqrt{q} \]

- **Lower bound:** from fluctuation of isolated edges in complement \(G_{n,q}\)
- **Upper bound:** from Talagrand’s inequality