Jump of the clique chromatic number of random graphs

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Parameter of Interest:

Clique Chromatic Nr. (χ_c) $\hat{=}$ Variant of Chr. Nr. with non-standard behavior

can "jump" by $n^{\Omega(1)}$ when $\varnothing$-degree of $G_{n,p}$ changes by $n^{o(1)}$

Question: Typical value of $\chi_c(G_{n,p})$ for Random Graph $G_{n,p}$?

This Talk: Solve Problem of McDiarmid, Mitsche, Prałat:

"Explain polynomial "jump" of $\chi(G_{n,p})$ near $p = n^{-1/2}$"

Motivation:

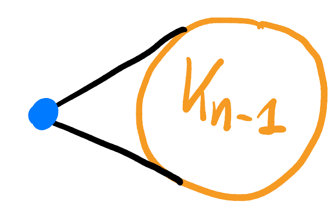
- Widely studied variant of Chromatic Nr.
- Testbed for probabilistic techniques ($\rightsquigarrow$ "beyond Janson's Inequality")

Clique Chromatic Nr

$\chi_c(G)$ = smallest # colors needed to vertex-color G s.t. no inclusion-maximal clique is monochromatic (ignoring isolated vertices)

Basic Properties

- arbitrary G : $\chi_c(G) \leq \chi(G)$
- Δ-free G : $\chi_c(G) = \chi(G)$
- $G = K_n$: $2 = \chi_c(K_n) < \chi(K_n) = n$



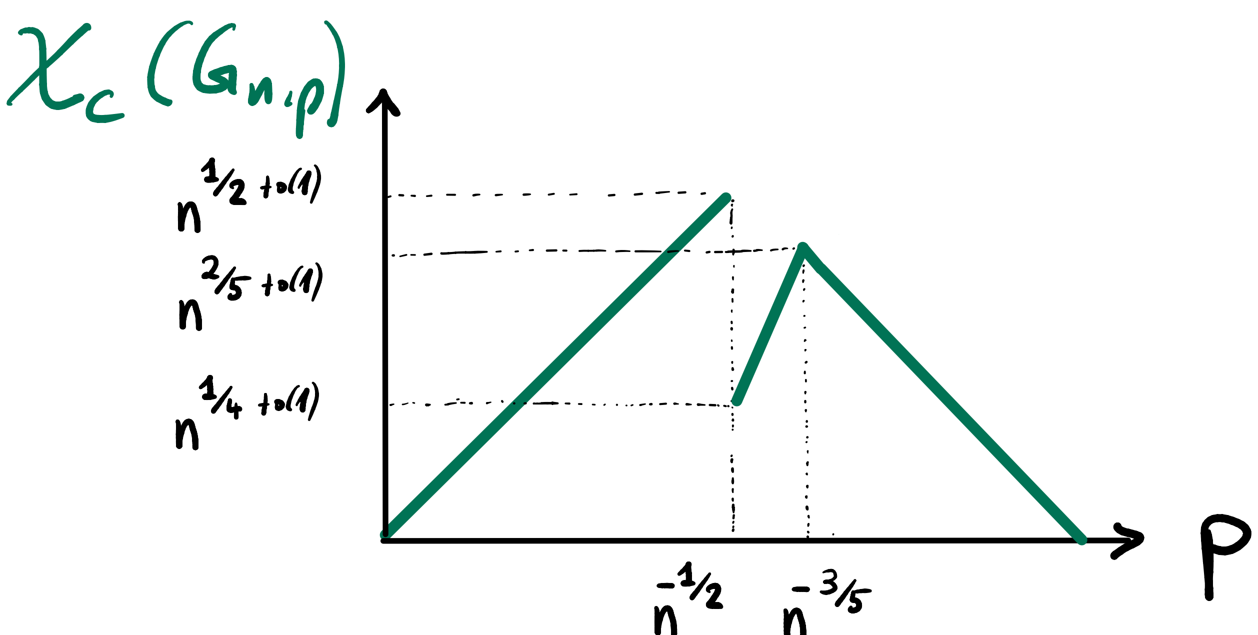
- not monotone : $\exists\ G \subseteq H$ with $\chi_c(G) > \chi_c(H)$

Quantity of Interest: $\chi_c(G_{n,p})$ = Clique Chr. Nr. of Random Graph $G_{n,p}$
(This Talk)

Clique Chromatic Nr

$\chi_c(G)$ = smallest # colors needed to vertex-color G s.t. no inclusion-maximal clique is monochromatic (ignoring isolated vertices)

Random Graph $G_{n,p}$ (McDiarmid, Mitsche, Prałat 2016)



Jump from $n^{1/2+o(1)}$ to $n^{1/4+o(1)}$ left widely open:

whp $n^{1/4+o(1)} \le \chi_c(G_{n,p}) \le n^{1/2+o(1)}$ for $p = n^{-1/2+o(1)}$

More concrete bounds:

If Ø-degree is $np = n^{x+o(1)}$ with $x \neq 1/2$, then whp $\chi_c(G_{n,p}) = n^{f(x)+o(1)}$ for function f that is discontinuous at $x = \frac{1}{2}$.

$\chi_c(G_{n,p})$ = Clique Chr. Nr. of Random Graph $G_{n,p}$

Main Result (Lyuben, Mitsche, W. 2021+)

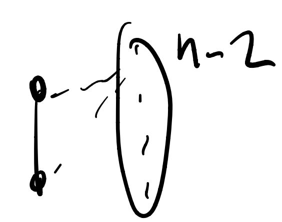
$$\text{whp} \quad \chi_c(G_{n,p}) = \tilde{\Theta}\left(\max\left\{\frac{e^{-np^2}np}{\log(np)}, \frac{p^{3/2}n}{\sqrt{\log n}}\right\}\right) \quad \text{for } p \in \left[n^{-1/2-\varepsilon}, n^{-1/2+\varepsilon}\right]$$

- Solves Open Problem due to McDiarmid, Mitsche, Pralat (2016)
- New e^{-np^2}-Factor explains polynomial "jump":

$$\chi_c(G_{n,p}) = \Theta\left(\frac{np}{\log(np)}\right) \quad \text{For } np^2 \ll 1 \qquad [\text{few edges in } \Delta\text{'s}]$$

$$\chi_c(G_{n,p}) = \Theta\left(\frac{p^{3/2}n}{\sqrt{\log n}}\right) \quad \text{For } np^2 \gg \log n \qquad [\text{all edges in } \Delta\text{'s}]$$

- $\mathbb{P}(\text{edge } e \text{ not in } \Delta) = (1-p^2)^{n-2} \approx e^{-np^2}$



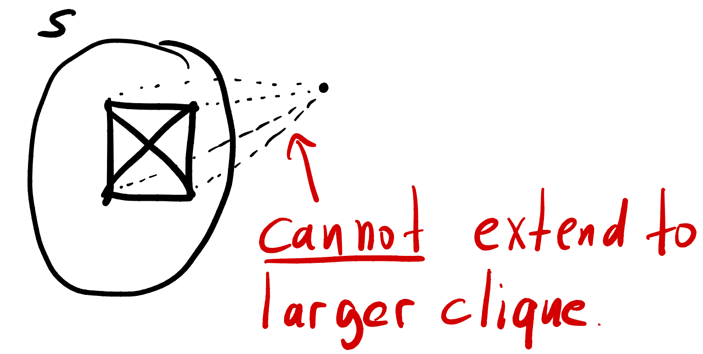
- Need to go beyond Janson's inequality to recover e^{-np^2}-Factor:

Proof-Strategy (Lower Bound)

① Basic Lower-Bound Observation:

If every s-vertex set contains maximal clique,
then every color class of χ_c has $\leq s$ vertices

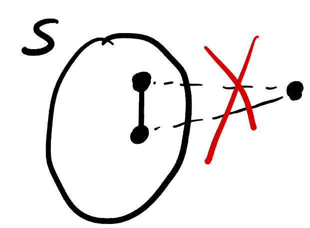
$$\Longrightarrow \quad \chi_c \geq \frac{n}{s}$$



② Implement for different s

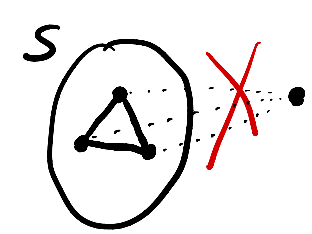
- $s = \dfrac{e^{np^2}\log(np)}{p}$ maximal edge

  $\hookrightarrow \quad \chi_c \geq \dfrac{n}{s} = \dfrac{e^{-np^2}\,np}{\log(np)}$



- $s = \dfrac{\sqrt{\log n}}{p^{3/2}}$ maximal Δ

  $\hookrightarrow \quad \chi_c \geq \dfrac{n}{s} = \dfrac{p^{3/2}\,n}{\sqrt{\log n}}$



Key-Difficulty:

Janson's Inequality
<u>not</u> good enough

Open Problem: Close logarithmic Gaps for $\chi(G_{n,p})$

$\chi_c(G_{n,p})$ = Clique Chr. Nr. of Random Graph $G_{n,p}$

Knowledge for $n^{-1/3+\varepsilon} \le p \ll 1$.

whp $\quad \frac{c}{p} \le \chi_c(G_{n,p}) \le C\frac{\log n}{p}$ $\quad$ for constants $c, C > 0$

every set of size $s = Cnp$ contains maximal clique K_k of size $k = \log_{\frac{1}{p}}(n)$

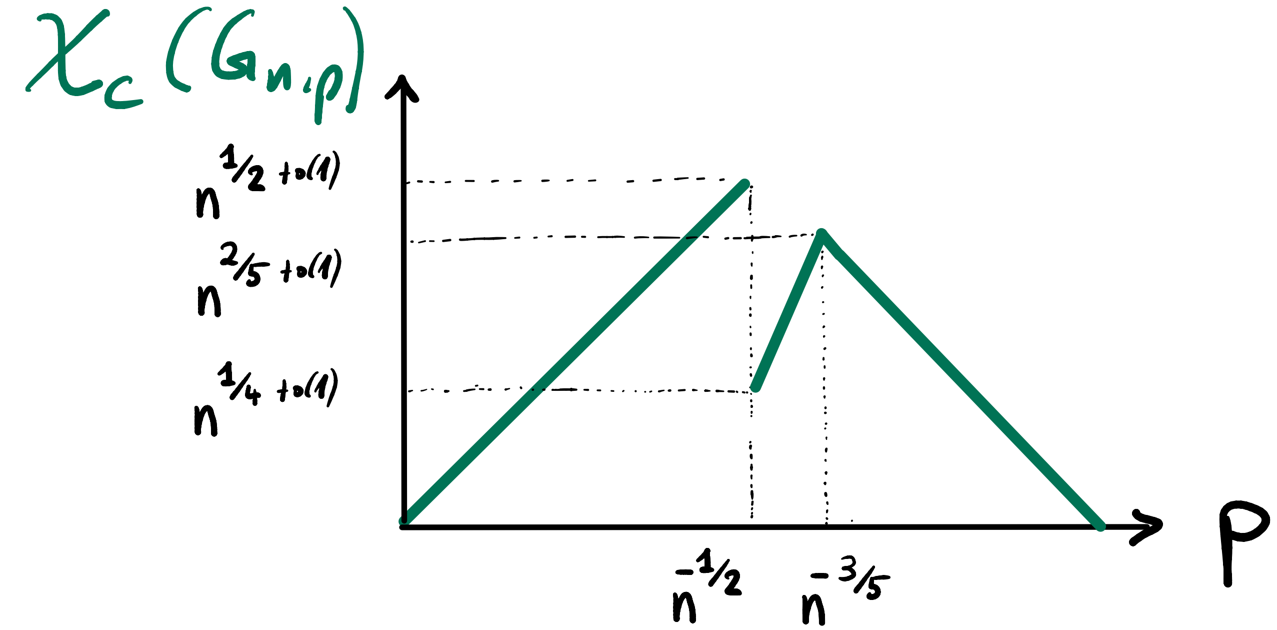
Connection to Domination-Nr.

Conjecture: ("upper bound gives correct order")

whp $\quad \chi_c(G_{n,p}) = \Theta\left(\frac{\log n}{p}\right)$ for $n^{-1/3+\varepsilon} \le p \ll 1$

Summary:

$\chi_c(G_{n,p})$ = Clique Chromatic Number of Random Graph $G_{n,p}$



Main Result (Lyuben, Mitsche, W. 2021+)

We explain "Jump" from $n^{1/2+o(1)}$ down to $n^{1/4+o(1)}$ near $p=n^{-1/2}$ via e^{-np^2}-factor

- Solves Problem due to McDiarmid, Mitsche, Pralat
- Proof needs to go beyond Janson's inequality
- Open Problem: order of $\chi_c(G_{n,p})$ for $n^{-1/3+\varepsilon} \leq p \ll 1$