The evolution of subcritical Achlioptas processes

Lutz Warnke
Georgia Tech

Joint work with Oliver Riordan

Context: classical Erdős–Rényi model

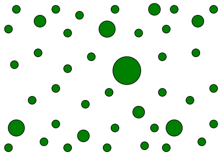
Erdős–Rényi random graph process

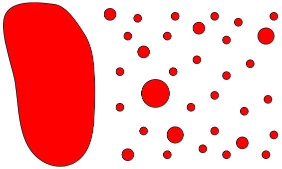
- Start with an empty graph on n vertices
- In each step: add a random edge to the graph

Phase transition of largest component (Erdős-Rényi, 1959)

Size 'dramatically changes' after $\approx n/2$ steps. For *fixed* t, whp

$$L_1(tn) = \begin{cases} \Theta(\log n) & \text{if } t < 1/2 \\ \Theta(n) & \text{if } t > 1/2 \end{cases}$$



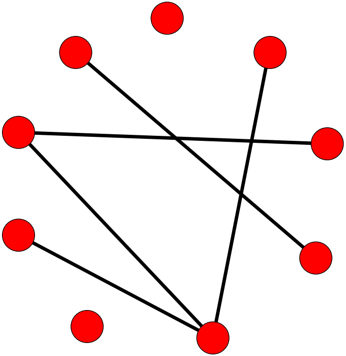


Variant of Erdős–Rényi with Dependencies

Achlioptas processes ('power of two random choices')

- Start with an empty graph on n vertices
- In each step: pick *two* edges uniformly at random (independently),
  add *one* of them to the graph (using some *rule*)

Dimitris Achlioptas



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Remarks:

- Yields *family* of random graph processes (includes Erdős–Rényi process)
- Interdisciplinary interest: ≥ 300 related papers since 2009
- *Key difficulty:* non-trivial dependencies between the edges added

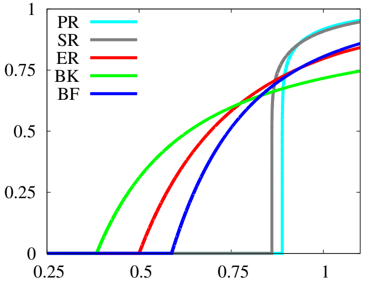
Motivation

- Improve our understanding of the phase transition phenomenon
- Test / develop methods for analyzing processes with dependencies

Behaviour of different Achlioptas proceses

Key example (suggested by Achlioptas)

Fraction of vertices in largest component after tn steps: $L_1(tn)/n$

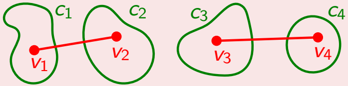


Goal of this talk

Understand/Analyze how these proceses *evolve over time*

Widely studied Achlioptas rules

Size rules



Decision (which edge to add) depends *only* on component *sizes* $c_1, \ldots, c_4$

- **Sum rule**: add $e_1 = \{v_1 v_2\}$ iff $c_1 + c_2 \leq c_3 + c_4$
  (‘add the edge which results in the smaller component’)

Bounded-size rules

All component sizes larger than some constant B are treated the same

- **Bohman–Frieze**: add $e_1 = \{v_1 v_2\}$ iff its endvertices are isolated
  (‘add random edge with slight bias towards joining isolated vertices’)

Previous work

Bounded-size rules: most things known

Phase transition of all *bounded-size rules* exhibits Erdős–Rényi behaviour

- Location of Phase-Transition (Spencer–Wormald, Riordan–W.)
- Critical Window (Bhamidi–Budhiraja–Wang)
- Sub- and Supercritical Phases (Riordan–W.)

Size rules: only one conditional result (Riordan–W.)

IF an associated system of differential equations has a *unique* solution, then key statistics (small/largest component) are *concentrated*

- System of differential equations can be *infinite* (e.g. for sum-rule)
- Uniqueness open question (but easy for bounded-size rules)

New Result for Size Rules

Susceptibility $\chi(G) = \sum_{k\geq 1} kN_k(G)/n$

- Expected size of component containing randomly selected vertex

New result for size rules (Riordan–W.)

Any size rule $\mathcal{R}$ is 'well-behaved' until a critical time $t_c = t_c^{\mathcal{R}}$,
where the *susceptibility* χ *diverges*. For *fixed* $t < t_c$, whp

- *Small components*: $N_k(tn) \sim \varrho_k^{\mathcal{R}}(t)n$
- *Exponential tails*: $N_k(tn) \leq Ae^{-ak}n \quad \rightarrow \quad L_1(tn) \leq \frac{2}{a}\log n$

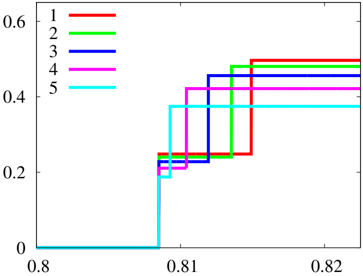
Conjecture for size rules (Riordan–W.)

For $t > t_c$ we have a giant component: whp $L_1(tn) = \Omega(n)$

- Motivated by percolation theory (equality of two critical point def.)
- True: bounded-size rules + certain size rules (e.g., max. sum rule)

A cautionary Example

Several simulations of $\varrho(t) = \frac{L_1(tn)}{n}$ using a certain size rule:

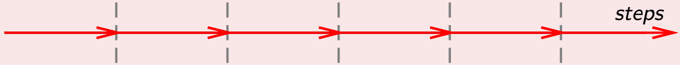


Punchline: Convergence up to t_c seems best possible

Beyond t_c some rules look *nonconvergent* in simulations

Structure of the proof

Inductively establish concentration (of a given size rule)



Need: evolution starting from initial graph F

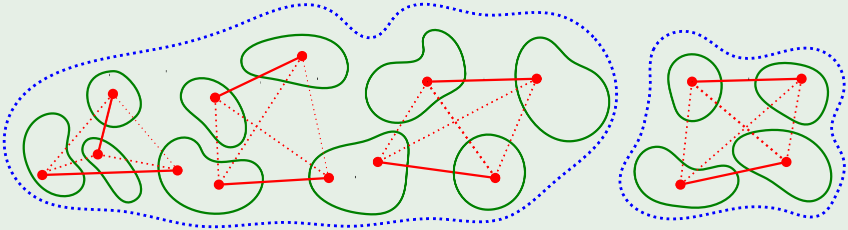
- Assumption: initial graph F is 'nice'
- Conclusion: graph after σn steps is again 'nice' (if σ *small enough*)

In comparison to bounded size rules

- We track key statistics *without* using differential equations
- We *investigate dependencies* amoung choices in more detail

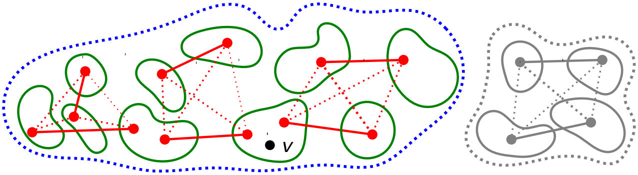
Investigating dependencies

How far can decisions propagate?



- For size rules, decisions can only propagate inside clusters
  - Here we *ignore order* of pairs
- Inside each cluster:
  - *Order of the pairs* uniquely determines decisions of any size rule

Glimpse of the proof



Determine component size $|C_v|$ via two-step exposure

- Reveal *all pairs of edges* offered
  - Determine relevant cluster for v $\approx$ Branching process
- Reveal *order* of all (relevant) pairs
  - Apply size rule $\mathcal{R}$ inside cluster

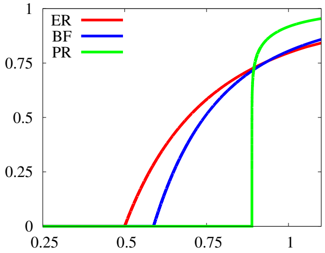
Why do we need susceptibility $\chi < \infty$?

- Branching process must be 'sub-critical' (need $\sigma \leq c\chi^{-1}$)
  - Only 'few' edges/components influence $|C_v|$ $\rightarrow$ Concentration

Summary

First rigorous result for size rules (Riordan–W.)

Key statistics are 'well-behaved' until the susceptibility diverges

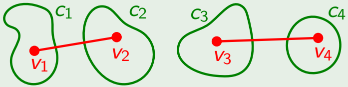


Open problems

- How can we analyze the later evolution of size rules?
- Does the phase transition occur when the susceptibility diverges?

The System of Differential Equations

Size rules decide using $c_1, \ldots, c_4$ only



$d_k(c_1, \ldots, c_4) =$ change of N_k given component sizes $c_1, \ldots, c_4$

Simplification: let's assume $v_1, \ldots, v_4$ are in *different* components

System of differential equations

Motivated by *expected one-step change* of $N_k(tn) \approx \varrho_k(t)n$:

$$\varrho'_k(t) = \sum_{c_1,\ldots,c_4 \in \mathbb{N} \cup \{\infty\}} d_k(c_1, \ldots, c_4) \prod_{j \in [4]} \varrho_{c_j}(t)$$