Lecture 3 – Tensors, Two Derivatives and (baby) Lie Groups **by L. Ni**

First we prepare some algebraic preliminaries for the later study of the differential geometry. More details can be found for example in Sakai's book.

Let V be a n-dimensional vector space. For M a smooth manifold, let T_pM be the tangent space at p, and let T_p^*M be the cotangent space, which is the dual of T_pM. In the discussion we identify V with T_pM, V^* the dual with T_p^*M. To understand the tensor fields we need to start with the construction of tensor products. If W is another linear vector space, then $V\otimes W$ is defined as the linear space spanned either linear maps $W^*\to V$ or as a bilinear form on $V^*\times W^*$ defined for any $v\in V, w\in W$ as below:

$$v\otimes w(w_1^*)=w_1^*(w)v;\quad v\otimes w(v_1^*,w_1^*)=v_1^*(v)w_1^*(w).$$

We also write $v^*(v_1)$ as $\langle v_1, v^*\rangle$ abusing the notation. Inductively the tensor product $V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^*$ can be defined. This is denoted as $T^r_s(V)$ if there are r components of V and s components of V^*. The tensor bundle $T^r_s(M)$ is defined to be the total space of $\cup_{p\in M}T^r_s(T_p(M))$. The space of smooth sections of these bundles are denoted by $\mathcal{T}^r_s(M)$. A linear isomorphism between vector spaces $\phi: V\to W$ naturally extends to the tensor products $T^r_s(V)\to T^r_s(W)$, which we denote by $\tilde{\phi}$.

Exterior product $\wedge^p(V)$ is defined as the skew symmetric multilinear forms on $V\times\cdots\times V$ as below for $x_1^*,\cdots,x_p^*\in V^*$ as

$$x_1^*\wedge\cdots\wedge x_p^*(x_1,\cdots,x_p)=\sum_\sigma \operatorname{sgn}(\sigma)x^*_{\sigma(1)}(x_1)\cdots x^*_{\sigma(p)}(x_p)=\det(x_i^*(x_j)).$$

Note that $\wedge^p(V)\subset T^0_p(V)$. Similarly we define $\wedge_p(V)$ as the skew symmetric multilinear forms on $V^*\times\cdots\times V^*$. The space $\wedge^k(T_pM)$ and $\wedge^k(M)=\cup_p\wedge^k(T_pM)$ is often used. The space of the smooth sections are called k-forms and denoted $\Omega^k(M)$. Clearly $\wedge^1(M)=T^*(M)$.

We use the convention that $\{e_i\}$ and $\{e^j\}$ are the basis of V and V^* dual to each other. If $\tilde{e}_i=a_i^k e_k$, it is easy to see that the dual $\tilde{e}^j=b^j_l e^l$ with $(b^j_l)=(a^k_i)^{-1}$. If $A:V\to W$ is an isomorphisms it is easy to see if we define $A^*:W^*\to V^*$ as $A^*(w^*)(v)=w^*(A(v))$, in terms of the basis A^* is the transpose of A. The induced map $V^*\to W^*$ would be $(A^*)^{-1}$.

In the case V is endowed with a metric $\{e_i\}$ and $\{e^j\}$ are assumed to be a orthonormal basis. The metric extends naturally to $T^r_s(V)$ and $\wedge^k(V)$. The convention is that $\{e_{i_1}\wedge\cdots\wedge e_{i_k}\}$ for all$_{1\le i_1<\cdots<i_k\le n}$ and $\{e^{j_1}\wedge\cdots\wedge e^{j_l}\}$ for all$_{1\le j_1<\cdots<j_l\le n}$ are orthonormal basis of $\wedge_k(V)$ and $\wedge^l(V)$. In particular,

$$\|x_1\wedge\cdots\wedge x_k\|^2=\det(\langle x_i,x_j\rangle).$$

Note that here Binet-Cauchy theorem yields a generalized Pythagorean theorem if we view the left hand side as the square of the k-dimensional volume of the parallelepiped generated by the vectors $\{x_1,\cdots,x_k\}$. (It says that such a square equals the sum of the squares of projections in all k-dimensional subspaces, with respect to any orthonormal basis.)

Secondly we consider two derivatives. Recall for any smooth function f, $df|_p$ is defined as the linear functional of T_pM acting as

$$df(v)=\left.\frac{d}{dt}\right|_{t=0} f(c(t)),\ \text{with } c(0)=p, \dot{c}(0)=v. \text{ This is also denoted as } L_v f.$$

In local coordinate $df = \frac{\partial f}{\partial x^i} dx^i$. The exterior differentiation extends to $\Omega^k(M)$ (in fact on germs of such) as follows: if $\omega = a dx^{i_1} \wedge \cdots dx^{i_k}$, $d\omega = da \wedge dx^{i_1} \wedge \cdots dx^{i_k}$. For general $\omega = a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots dx^{i_k}$ it extends linearly. It then can be easily checked that $\omega = \alpha \wedge \beta$, $d\omega = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ if $\alpha \in \Omega^k(M)$.

Another important derivative is the Lie derivative on tensors (in particular for forms). Given a vector field X for any smooth function $f(x)$, $(L_X f)(x) \doteqdot L_{X(x)} f$. By ODE, X generated a one parameter family diffeomorphisms $\varphi(t,p)$ (also denoted by φ_t when there is no confusion) which satisfies that $\varphi(0,p) = p$. It then induces the map between the germs of smooth functions $d\tilde{\varphi}_{-t} : \mathcal{F}_{\varphi_t(p)} \to \mathcal{F}_p$ by the $d\tilde{\varphi}_{-t}(f)(x) = f(\varphi_t(x))$. Clearly $(Xf)(p) = \frac{d}{dt}|_{t=0} d\tilde{\varphi}_{-t}(f)$. Since $d\tilde{\varphi}_{-t}$ can be extended to $T_{\varphi_t(p)}M \to T_pM$ as $(d\varphi_t)^{-1} = d\varphi_{-t}$ and then to $T^t_s(T_{\varphi_t(p)}M) \to T^r_s(T_pM)$. Similarly the Lie derivative $L_X T$ is defined as

$$L_X T = \frac{d}{dt} d\tilde{\varphi}_{-t}(T)\Big|_{t=0}.$$

It can be easily checked that this derivative is compatible with the contraction and satisfies the Leibniz rule. In fact if $T = Y \otimes Z^*$, $L_X(C^1_1(T))|_p = X(\langle Y, Z^* \rangle)|_p$ which equals to

$$I = \lim_{t\to 0} \frac{1}{t} \left(\langle Y_{\varphi_t(p)}, Z^*_{\varphi_t(p)} \rangle - \langle Y_p, Z^*_p \rangle \right) = \lim_{t\to 0} \frac{1}{t} \langle (d\tilde{\varphi}_{-t}(Y_{\varphi_t(p)}), d\tilde{\varphi}_{-t}(Z^*_{\varphi_t(p)}) \rangle - \langle Y_p, Z^*_p \rangle.$$

One the other hand $C^1_1(L_X(T)) = C^1_1(L_X Y \otimes Z^* + Y \otimes L_X Z^*) = \langle L_X Y, Z^* \rangle + \langle Y, L_X Z^* \rangle$. The first term can be written as

$$\langle L_X Y, Z^* \rangle = \lim_{t\to 0} \frac{1}{t} \langle d\tilde{\varphi}_{-t}(Y_{\varphi_t(p)}) - Y_p, Z^*_p \rangle = I + \lim_{t\to 0} \frac{1}{t} \langle d\tilde{\varphi}_{-t}(Y_{\varphi_t(p)}), Z^*_p - d\tilde{\varphi}_{-t}(Z^*_{\varphi_t(p)}) \rangle.$$

The second term above equals $-C^1_1(Y \otimes L_X Z^*)$, hence this proves the claim that the Lie derivative is commutative with the contraction.

Lemma 0.1. *(i) For any vector fields X, Y then $L_X Y = [X, Y]$. (ii) For any $\omega \in \Omega^p(M)$ (or a local germ),*

$$L_X = d \circ \iota_X + \iota_X \circ d. \tag{1}$$

Here $\iota_X : \wedge^{k+1}(V) \to \wedge^k(V)$ is defined as $\iota_X(\omega)(X_1, \cdots, X_k) = \omega(X, X_1, \cdots, X_k)$. Since $x^*_1 \wedge \cdots \wedge x^*_k = \sum_\sigma \mathrm{sgn}(\sigma) x^*_{\sigma(1)} \otimes \cdots \otimes x^*_{\sigma(k)}$, ι_X can be obtained from $\iota_X : T^0_k(V) \to T^0_{k-1}$ defined as $\iota_X(T) = C^1_1(X \otimes T)$, where C^1_1 is the contraction operator (at position $(1,1)$).

Proof. (i). Let φ_t, ψ_s be the group generated by X and Y respectively. For any smooth f we have $[X,Y]f = XYf - YXf$. Hence

$$(XYf)_p - (YXf)_p = \frac{\partial}{\partial t} \frac{\partial}{\partial s} \left[f(\psi_s(\varphi_t(p))) - f(\varphi_t(\psi_s(p))) \right] \Big|_{t=0, s=0}.$$

On the other hand $(L_X Y)(q) = \lim_{t\to 0} \frac{1}{t} \left[d\varphi_{-t}(Y(\varphi_t(q))) - Y(q) \right]$, which however also equals to $\lim_{t\to 0} \frac{1}{t} \left[Y(\varphi_t(q)) - d\varphi_t(Y(q)) \right]$. Hence

$$(L_X Y)f = \lim_{t\to 0} \frac{1}{t} \left[\frac{\partial}{\partial s}\Big|_{s=0} \left(f(\psi_s(\varphi_t(p))) - f(\varphi_t(\psi_s(p))) \right) \right] \tag{2}$$

which proves the claimed result.

To prove (ii), it suffice to check that $P_X \doteqdot d \circ \iota_X + \iota_X \circ d$ is a *derivation* and it equals L_X on the 1-forms, in fact sufficiently on $\omega = dx^i$. Both of them can be checked via direct calculations. In fact $(L_X\omega)(Y) = X(\omega(Y)) - \omega(L_X Y) = X(Y^i) - [X,Y]^i = X^l \frac{\partial Y^i}{\partial x^l} - \left(X^l \frac{\partial Y^i}{\partial x^l} - Y^l \frac{\partial X^i}{\partial x^l}\right) = Y^l \frac{\partial X^i}{\partial x^l}$. On the other hand for $\omega = dx^i$, $[d \circ \iota_X + \iota_X \circ d(\omega)](Y) = Y(\omega(X)) = Y^l \frac{\partial X^i}{\partial x^l}$.

To check that P_X is a deviation we need to check (a) $P_X(f\omega) = X(f)\omega + fP_X(\omega)$ and (b) $P_X(\alpha \wedge \beta) = P_X(\alpha) \wedge \beta + \alpha \wedge P_X(\beta)$. For (a), it is easy to check that $\iota_X(df \wedge \omega) = X(f)\omega - df \wedge \iota_X\omega$. This implies (a) by easy computations. For (b) we extends the above identity to $\iota_X(\alpha \wedge \beta) = (\iota_X(\alpha)) \wedge \beta + (-1)^k \alpha \wedge \iota_X(\beta)$ for $\alpha \in \wedge^k(V)$ and $\beta \in \wedge^l(V)$. It suffices to verify this identity for $\alpha = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\beta = dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_{k+l}}$, which can be done with the following observation:

$$\iota_X(\alpha \wedge \beta) = \sum_{t=1}^{k+l} (-1)^{t-1} X(x^{i_t}) dx^{i_1} \wedge \cdots \hat{dx^{i_t}} \wedge dx^{i_{k+l}}.$$

The claim (b) then follows by straight forward calculations. $\square$

Corollary 0.1. *(i) ψ_s and φ_t is commutative if and only if $[X,Y] = 0$. (ii) For $\omega \in \Omega^k(M)$ and vector fields $X_0, \cdots, X_k$,*

$$\begin{aligned} (3) \quad d\omega(X_0, \cdots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \cdots, \hat{X}_i, \cdots, X_k)) \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k). \end{aligned}$$

Proof. The part (i) follows from the above proof, the part (ii) can be done via the induction and we leave this as an exercise. In fact, by the proof (precisely (2)) if φ_t commutes with ψ_s it is easy to see that $L_X Y f = 0$ for any f, hance $L_X Y = 0$. On the other hand, if $L_X Y = 0$, $\frac{d}{dt}(d\varphi_{-t}(Y(\varphi_t(q)))) = 0$. Hence $d\varphi_{-t}(Y(\varphi_t(q))) = Y(q)$ for any q. This implies that $\varphi_{-t}(\psi_s(\varphi_t(q))) = \psi_s(q)$ by the uniqueness of the integrations of Y.

For part (ii), clearly the result holds for $k = 0$. For $k = 1$, $d\omega(X_0, X_1) = \iota_{X_0} d\omega(X_1) = (L_{X_0}\omega)(X_1) - (d(\iota_{X_0}\omega))(X_1) = X_0(\omega(X_1)) - \omega(L_{X_0} X_1) - X_1(\omega(X_0))$. Inductively,

$$\begin{aligned} d\omega(X_0, X_1, \cdots, X_k) &= (\iota_{X_0} d\omega)(X_1, \cdots, X_k) = (L_{X_0}\omega - d \circ \iota_{X_0}(\omega))(X_1, \cdots, X_k) \\ &= X_0(\omega(X_1, \cdots, X_k)) - \sum_{j=1}^{k} \omega(X_1, \cdots, [X_0, X_j], \cdots, X_k) \\ &\quad - \sum_{i=1}^{k} (-1)^{i-1} X_i(\iota_{X_0}\omega(X_1, \cdots, \hat{X}_i, \cdots, X_k)) \\ &\quad - \sum_{1 \le i < j \le k} (-1)^{i+j} \iota_{X_0}\omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots \hat{X}_j, \cdots, X_k). \end{aligned}$$

Rewrite some of the terms and collect them properly we have the claim. $\square$

Thirdly we include some very basics for Lie groups. A Lie group G is defined to be a manifold with smooth group structure. By the definition the map $G \times G \to G$ defined as $(a, b) \to ab^{-1}$ is smooth. Hence $L_a(g) = ag$ and $R_a(g) = ga$ are smooth mappings. For any $X_a \in T_eG$, $dL_a(X_e)$ defines a smooth vector field on G which is *left invariant*. Then introduce the Lie algebra structure $[x, y] = [X, Y]$ where $x, y \in T_eG$ and X, Y are the left invariant extensions of x, y. T_eG endowed with this structure is called Lie algebra of G, which is denoted by $\mathfrak{g}$. If X is a left invariant vector field, let $\varphi_t(p)$ be the integration of X and $\varphi_t(e)$ be the 1-parameter subgroup. (Due to that $\varphi_t(a) = a\varphi_t(e)$.) We also write $\varphi_t(e) = \exp(tX)$. (Exponential map for Lie groups). There is a nature representation of G as a linear transformation of $\mathfrak{g}$. For a define $\mathrm{Ad}_G(g) = aga^{-1}$. Then $d(\mathrm{Ad}_G)|_e$ is a general linear transformation of $\mathfrak{g}$, which we denote as $\mathrm{Ad}_{\mathfrak{g}}$. Conventionally we write it as $\mathrm{Ad} : G \to GL(\mathfrak{g})$. This is called the *adjoint representation* of G. Its differential at T_eG is denoted by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$.

Lemma 0.2. *For X, Y, left invariant vector fields.*

$$[X, Y]_e = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\exp(tX))(Y_e).$$

Namely $\mathrm{ad}(x)(y) = [x, y]$.

Proof. Note $\varphi_t(p) = R_{\exp(tX)}(p)$. By the previous lemma $[X, Y]_e = \frac{d}{dt}|_{t=0}(d\varphi_{-t}(Y_{\varphi_t(p)}) = \frac{d}{dt}|_{t=0} dR_{\exp(-tX)}(Y_{\varphi_t(p)}) = \frac{d}{dt}|_{t=0} dR_{\exp(-tX)} dL_{\exp(tX)}(Y_e) = \frac{d}{dt}|_{t=0}\mathrm{Ad}(\exp(tX))(Y_e)$, which by the definition equals to $\mathrm{ad}(X_e)(Y_e)$. □

Lemma 0.3. *If $\rho : G_1 \to G_2$ is a Lie group homeomorphism*

$$\rho(\exp(X)) = \exp(d\rho(X)).$$

Proof. Observe that $\rho(\exp(tX))$ is a 1-parameter sub-group with tangent being $d\rho(X)$ at e. Hence the result follows. □

Corollary 0.2. *Applying the above to* $\mathrm{Ad} : G \to GL(\mathfrak{g})$ *we have*

$$\mathrm{Ad}_G(\exp(tX)) = \exp(t\mathrm{ad}(X)). \tag{4}$$

Exercises: 1. Let $\varphi_t(p)$ and $\psi_s(p)$ be the integrations of vector fields X, Y. Prove that the tangent of $\psi_{-\sqrt{t}}(\varphi_{-\sqrt{t}}(\psi_{\sqrt{t}}(\varphi_{\sqrt{t}}(p))))$ is $[X, Y]$.

2. Let φ_t and ψ_s be the one parameter family diffeomorphisms generated by X and Y. Define $g_1(c) = \varphi_c(p)$, $g_2(c) = \psi_c(g_1(c))$, $g_3(c) = \varphi_{-c}(g_2(c))$ and $g_4(c) = \psi_{-c}(g_3(c))$. Let $h(t) = g_4(t)$ and $g(t^2) = h(t)$. Then for any smooth germ f

$$[X, Y]_p(f) = \lim_{t\to 0^+} \frac{f(g(t)) - f(p)}{t} = \frac{1}{2}(f(h(t))''(0).$$

3. For $X, Y \in \mathfrak{g}$, prove $\exp(ts[X, Y]) = \exp(sX)\exp(tY)\exp(-sX)\exp(-tY) + o(st)$. Hence if $\tilde{X}$, $\tilde{Y}$ are vector fields on M induced by $\varphi_t = \exp(tX)$ and $\psi_t = \exp(tY)$, by exercise 1 $[\tilde{X}, \tilde{Y}]$ is induced by $-[X, Y]$.

Lecture 4 – Connexion, Holonomy and Covariant Derivatives by L. Ni

A fiber bundle is a triple (E, F, M) with a projection map $p : E \to M$ such that p is regular with $p^{-1}(x)$ being diffeomorphic to F such that for any point $p \in M$, there exists U_α and $\varphi : U_\alpha \times F \to p^{-1}(U_\alpha)$ such that $\varphi(x, f) \in p^{-1}(x)$. We say it has a *structure group* G, if the transition functions $T_{\alpha\beta}(x)$ (where $\varphi_\beta^{-1} \circ \varphi_\alpha(x, f) = (x, T_{\alpha\beta}(x)(f))$ is in G. For example $(TM, \mathbb{R}^n, M)$ is the tangent bundle with the transition group being $GL(\mathbb{R}^n)$. If M is endowed with a Riemannian metric $(TM, \mathbb{R}^n, M)$ can have a structure group of $\mathsf{O}(n)$. For vector bundles of dimension k we can define its tensor bundles and exterior product bundles (the structure group will be the corresponding representation of $GL(\mathbb{R}^n)$ or $\mathsf{O}(n)$ acting on the corresponding tensor products). This general formulation admits a lot more cases. One example is $(\widetilde{M}, F, M)$ with $\widetilde{M}$ being a covering of M, F being the discrete set of isolated points can be viewed as a very special case. For this the structure group is $\sigma(\pi_1(M))$, the monodromy group of the covering (with $\sigma : \pi_1(M) \to$ the permutation group of the discrete sets F). Another is the *principle bundle*, with F being a Lie group G with the structure group being G acting by the left multiplication.

A connexion of (E, F, M) is for any piece-wisely smooth path $\gamma : (0, 1) \to M$, there exists $\varphi_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ such that it satisfies that φ depends on γ smoothly, and $\varphi_{\gamma_1 \circ \gamma_2} = \varphi_{\gamma_1} \circ \varphi_{\gamma_2}$ and $\varphi_{\gamma^{-1}} = (\varphi_\gamma)^{-1}$. Such φ_γ is called the *parallel transport* along γ. In general φ is in $\mathrm{Diff}(F)$, but in specific cases it lies inside the structure group G correspondingly.

These concepts can be pushed further into the case of that (E, M) being topological spaces and p being continuous and having path-lifting and covering homotopy properties (called a *Serre fibration*). There one has correspondingly concept of a *homotopy connexion*, which is a homotopy equivalence of the fibers $p^{-1}(\gamma(0))$ and $p^{-1}(\gamma(1))$. E.g. the covering spaces are ones where the homotopy connexion is defined by lifting pathes.

Let $\Omega(x_0, M)$ be the *loop spaces* at x_0. Then φ_γ is a homomorphism $\varphi : \Omega(x_0, M) \to \mathrm{Diff}(F_{x_0})$ (or G). Then the image (denoted by H_{x_0}) is called the *holonomy group.* For covering spaces, this is just the monodromy (namely the representation of $\pi_1(M)$ in the permutation of the fiber). For most discussion emphasizes are given to the image of the connected component of the trivial loop $\gamma(t) \equiv x_0$, namely the loops which are homotopically trivial. This is called the relative holonomy group, denoted by $H^0_{x_0}$. It is easy to see that for a different choice of the base point x_1, if γ is a path from x_0 to x_1, then $H_{x_0} = \varphi_{\gamma^{-1}} H_{x_1} \varphi_\gamma$.

A *covariant derivative* at point p is a map $\nabla : T_pM \times \mathcal{T}_pM \to T_pM$ ($\mathcal{T}_pM$ germs of tangent vectors) satisfying axioms: (i) $\nabla_{\alpha\xi+\beta\eta}Y = \alpha\nabla_\xi Y + \beta\nabla_\eta Y$; (ii) linear in the second component; (iii) $\nabla_\xi(fY) = (\xi f)Y + f\nabla_\xi Y$. This is also called an *affine connection.* The covariant derivative is a concept more linear than the Lie derivative since for smooth vectors X, Y and function f, $\nabla_{fX}Y = f\nabla_X Y$, a property fails to hold for the Lie derivative. A *global affine connection* is the one defined for all $p \in M$ satisfying that if X, Y are smooth $\nabla_X Y$ is smooth. Once M is endowed with a global affine connection we can define the *covariant derivative along a curve* $c(t) : (a, b) \to M$ (even along a smooth map $\phi, N \to M$) for a *vector field* $X(t)$ along $c(t)$ by $\frac{D}{dt}X(c(t)) = \nabla_{\dot{c}(t)}X$, if X is defined globally near $c(t)$. This leads to the connexion defined above via the concept of *parallel transport* along $c(t)$. For any $X_{x_0} \in T_{x_0}M$ and a curve $\gamma(t)$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, $X(t) \in T_{\gamma(t)}M$ can be constructed by solving the ODE $\frac{D}{dt}X(t) = 0$. Then define $\varphi_\gamma(X(0)) = X(1)$. We extends the definition to $\varphi_\gamma^{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$ as $\varphi_\gamma^{t_1,t_2}(\xi) = X(t_2)$ if X is parallel with $X(t_1) = \xi$. Note that the above discussion makes sense for a smooth vector bundle (E, M) of rank k. A basic result below asserts that a connexion on a vector bundle (with linear structure group) is equivalent to an affine connection.

Lemma 0.1.
$$\frac{D}{dt}X(t)\bigg|_{t_0} = \lim_{t\to t_0}\frac{\varphi_\gamma^{t,t_0}(X(t)) - X(t_0)}{t-t_0}.$$

As before we can extends the covariant derivative to the whole tensor spaces $\mathcal{T}_s^r(M)$ and show that it preserves the type and commutes with the contraction. Once there exists an affine connection one can define the geodesics by requiring the curve $c(t)$ satisfies $\frac{D\dot{c}}{dt} = 0$. Note that the concept of the geodesic is a bit nonlinear, which only makes sense for affine connections on TM.

On a Riemannian manifold, there exists a canonical affine connection called *Levi-Civita* connection ∇. A Levi-Civita satisfies two more requirements. (i) It is torsion free (namely for any smooth vector fields X, Y, $[X, Y] = \nabla_X Y - \nabla_Y X$); (ii) and it is compatible with the metric (namely $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Y Z\rangle$). The induced connexion of a metric compatible connection is the parallel transports in $\mathsf{O}(n)$.

Lemma 0.2. *Let ∇ be a torsion free connection. For $\omega \in \Omega^k(M)$,*

$$d\omega(X_0, X_1, \cdots, X_k) = \sum_{i=0}^{k}(-1)^i(\nabla_{X_i}\omega)(X_0, \cdots, \hat{X}_i, \cdots, X_k). \tag{1}$$

Proof. Note that the right hand side above, using that the covariant derivative is commuting with the contraction, can be written as

$$\sum_{i=0}^{k}(-1)^i\left[X_i(\omega(X_0, \cdots, \hat{X}_i, \cdots X_k) - \sum_j \omega(X_0, \cdots \hat{X}_i, \cdots, \nabla_{X_i}X_j, \cdots, X_k)\right]$$

where the second term the summation is for $j \neq i, 0 \leq j \leq k$ and the $\nabla_{X_i}X_j$ can appear before the i-th term.

On the other hand, by Corollary 1 of last lecture we know that first summand of the right hand side (in Corollary 1) matches the first term above. The second summand can be written as

$$\begin{aligned} & \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega(\nabla_{X_i}X_j - \nabla_{X_j}X_i, X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k) \\ = \quad & \sum_{i=0}^{k}(-1)^{i-1}\sum_{j>i}\omega(X_0, \cdots, \hat{X}_i, \cdots, \nabla_{X_i}X_j, \cdots, X_k) \\ & +\sum_{i=0}^{k}(-1)^{j-1}\sum_{j>i}\omega(X_0, \cdots, \nabla_{X_j}X_i, \cdots, \hat{X}_j, \cdots, X_k). \end{aligned}$$

Putting them together we have the claim. $\square$

One may view the right hand side as a derivative d_∇ induced by the covariant derivative on the forms. We remark that equation (1) holds for any 1-forms ω implies that ∇ is torsion free. Namely

$$\begin{aligned} 0 &= (d_\nabla - d)\alpha(X,Y) = \nabla_X\alpha(Y) - \nabla_Y\alpha(X) - d\alpha(X,Y) \\ &= X(\alpha(Y)) - \alpha(\nabla_X Y) - Y(\alpha(X)) + \alpha(\nabla_Y X) - X(\alpha(Y)) + Y(\alpha(X)) + \alpha([X,Y]) \\ &= \alpha([X,Y] - \nabla_X Y + \nabla_Y X). \end{aligned}$$

This implies the connection is torsion free. The lemma simply states that it coincides with the exterior derivative if the affine connection is torsion free.

The torsion of an affine connection is defined as $T(x,y) = -(\nabla_X Y - \nabla_Y X - [X,Y])$, with X, Y being the extension of $x, y \in T_pM$, which can be easily checked to be a tensor and skew-symmetric. The curvature tensor R is defined to be for any $x, y, z \in T_pM$, $\mathrm{R}_{x,y}z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$ with X, Y, Z being extensions of x, y, z. Generally for vector bundle (E, M) we may denote D as the connection and define similarly $\mathrm{R}_{x,y}s = -D_X D_Y s + D_Y D_X s + D_{[X,Y]}s$ and check that it does not depend on the extension of s. It can be checked that $\mathrm{R}_{x,y}s = -\mathrm{R}_{y,x}s$. Both can be defined for connexions of principal bundles (which probably goes back to Cartan, see the paper of Ambrose-Singer).

If ∇ is torsion free on TM then R satisfies the 1-*st Bianchi identity*:

$$\mathrm{R}_{x,y}z + \mathrm{R}_{y,z}x + \mathrm{R}_{z,x}y = 0. \tag{2}$$

In fact for X, Y, Z extensions of x, y, z we have

$$\begin{aligned} \mathrm{R}_{x,y}z + \mathrm{R}_{y,z}x + \mathrm{R}_{z,x}y &= -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z - \nabla_Y\nabla_Z X + \nabla_Z\nabla_Y X \\ &\quad +\nabla_{[Y,Z]}X - \nabla_Z\nabla_X Y + \nabla_X\nabla_Z Y + \nabla_{[Z,X]}Y \\ &= -\nabla_X[Y,Z] + \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_{[X,Y]}Z \\ &\quad +\nabla_{[Y,Z]}X + \nabla_{[Z,X]}Y \\ &= [[Y,Z],X] + [[Z,X],Y] + [[X,Y],Z] = 0. \end{aligned}$$

Note that this and the above lemma on the exterior derivative only make sense for affine connections on TM the tangent bundle.

The next two results however holds for general affine connections on vector bundles. Note that R can be viewed as a section of $\wedge^2(M) \otimes E^* \otimes E$. Then $D_x R_{y,z}$ can be defined if there exists ∇ on TM.

The curvature of a torsion free affine connection also satisfies the 2-*nd Bianchi idenity*:

$$D_x\mathrm{R}_{y,z} + D_y\mathrm{R}_{z,x} + D_z\mathrm{R}_{x,y} = 0. \tag{3}$$

The proof (we only present for the case $E = TM$) is quite similar to the above. Note that $\nabla_x\mathrm{R}_{y,z}$ is defined independent of the extensions X, Y, Z and W. Precisely

$$\begin{aligned} (D_X\mathrm{R}_{Y,Z})W &= D_X(\mathrm{R}_{Y,Z}W) - \mathrm{R}_{Y,Z}(D_X W) - \mathrm{R}_{\nabla_X Y,Z}W - \mathrm{R}_{Y,\nabla_X Z}W \\ &= D_X(-D_Y D_Z W + D_Z D_Y W) - (-D_Y D_Z D_X W + D_Z D_Y D_X W) \\ &\quad +D_X D_{[Y,Z]}W - D_{[Y,Z]}D_X W - \mathrm{R}_{\nabla_X Y,Z}W - \mathrm{R}_{Y,\nabla_X Z}W. \end{aligned}$$

The claimed result follows by summing the three equations obtained by permuting X, Y, Z above. Note that $D_X D_{[Y,Z]}W - D_{[Y,Z]}D_X W + D_Y D_{[Z,X]}W - D_{[Z,X]}D_Y W + D_Z D_{[X,Y]}W - D_{[X,Y]}D_Z W = -(\mathrm{R}_{X,[Y,Z]} + \mathrm{R}_{Y,[Z,X]} + \mathrm{R}_{Z,[X,Y]})W + D_{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]}W$, which equals to $-(\mathrm{R}_{X,[Y,Z]} + \mathrm{R}_{Y,[Z,X]} + \mathrm{R}_{Z,[X,Y]})W$, by the Jacobi identity on $[\cdot,\cdot]$.

Note that the proof works if D is an affine connection on a vector bundle E. This can also be obtained via the exterior derivative d_D defined on differential forms valued in E (See L+4). Next is a useful commutator formula on any tensors.

For any $T \in \mathcal{T}^r_s(M)$ we define invariantly the second derivative

$$\begin{aligned}
\nabla^2_{Y,X}T(\cdot) &\doteqdot \nabla^2 T(\cdot, X, Y) = \nabla_Y(\nabla T)(\cdot, X) \\
&= Y(\nabla T(\cdot, X)) - \nabla T(\nabla_Y(\cdot), X) - \nabla T(\cdot, \nabla_Y X) \\
&= Y(\nabla_X T(\cdot)) - \nabla_X T(\nabla_Y(\cdot)) - \nabla_{\nabla_Y X} T(\cdot) = \nabla_Y \nabla_X T(\cdot) - \nabla_{\nabla_Y X} T(\cdot).
\end{aligned}$$

Lemma 0.3. *Let $T \in \mathcal{T}^r_s(M)$ and ∇ be a torsion free affine connection. Then*

$$\nabla^2_{Y,X}T - \nabla^2_{X,Y}T = \mathrm{R}_{X,Y} \circ T, \ \textit{equivalently}\ \mathrm{R}_{X,Y}(T) = \mathrm{R}_{X,Y} \circ T, \tag{4}$$

where $\mathrm{R}_{X,Y}(T)(\cdot) = -\nabla_X\nabla_Y T(\cdot) + \nabla_Y \nabla_X T(\cdot) + \nabla_{[X,Y]}T(\cdot)$, *and* $\mathrm{R}_{X,Y}\circ$ *means that* $\mathrm{R}_{X,Y} : T_pM \to T_pM$ *acts on the tensor product as an algebraic derivation.*

Proof. First for $W^* \in \mathcal{T}^0_1$, $(\nabla_X W^*)(Z) = X(W^*(Z)) - W^*(\nabla_X Z)$, Hence

$$\begin{aligned}
(\nabla_Y \nabla_X W^*)(Z) &= Y((\nabla_X W^*)(Z)) - (\nabla_X W^*)(\nabla_Y Z) \\
&= YX(W^*(Z)) - Y(W^*(\nabla_X Z)) - X(W^*(\nabla_Y Z)) + W^*(\nabla_X \nabla_Y Z).
\end{aligned}$$

Hence if extends X, Y with $[X, Y] = 0$ we have that

$$(\nabla_Y \nabla_X W^*)(Z) - (\nabla_X \nabla_Y W^*)(Z) = -W^*(R_{X,Y}Z) = \left[-(R_{X,Y})^*(W^*)\right](Z).$$

Namely $\nabla^2_{Y,X}W^* - \nabla^2_{X,Y}W^* = \nabla_Y\nabla_X W^* - \nabla_X \nabla_Y W^* = -(\mathrm{R}_{X,Y})^*(W^*)$. This proves the result for the special case $T \in \mathcal{T}^0_1$. The general case follows by a similar argument. Precisely

$$\begin{aligned}
&(\nabla_Y(\nabla_X T))(W_1^*, \cdots, W_r^*, X_1, \cdots, X_s) = YX(T(W_1^*, \cdots, W_r^*, X_1, \cdots, X_s)) \\
&- \sum_{i=1}^r Y(T(W_1^*, \cdots, \nabla_X W_i^*, \cdots, W_r^*, X_{\dots})) - \sum_{k=1}^s Y(T(W_{\dots}^*, X_1, \cdots, \nabla_X X_k, \cdots, X_s)) \\
&- \sum_{i=1}^r X(T(W_1^*, \cdots, \nabla_Y W_i^*, \cdots, W_r^*, X_{\dots})) - \sum_{k=1}^s X(T(W_{\dots}^*, X_1, \cdots, \nabla_Y X_k, \cdots, X_s)) \\
&+ \sum_{i \neq j} T(W_1^*, \cdots, \nabla_X W_i^*, \cdots, \nabla_Y W_j^*, \cdots, W_k^*, X_{\dots}) \\
&+ \sum_{k \neq l} T(W_{\dots}^*, X_1, \cdots, \nabla_X X_k, \cdots, \nabla_Y X_l, \cdots, X_s) \\
&+ \sum_{i=1}^r T(W_1^*, \cdots, \nabla_X \nabla_Y W_i^*, \cdots, W_r^*, X_{\dots}) + \sum_{k=1}^s T(W_{\dots}^*, X_1, \cdots, \nabla_X \nabla_Y X_k, \cdots, X_s)
\end{aligned}$$

The claimed result follows by subtracting from the above the same equation with X, Y swapped. □

Another proof can be done by assuming $T = X_1 \otimes \cdots \otimes X_r \otimes W_1^* \otimes \cdots \otimes W_s^*$ and applying $\nabla_Y \nabla_X - \nabla_X \nabla_Y$ into via the Leibniz rule. Once M is endowed with a metric we define $\Delta T = \nabla^2_{e_i, e_i} T$ for an othonormal frame $\{e_i\}$. Note that this lemma applies to the affine connections on any rank k vector bundle $(E, \mathbb{R}^k, M)$ over the manifold M. To make it sensible a torsion free connection ∇ on TM and an affine connection $D : T_pM \times \Gamma(E) \to E_p$ are needed. The second derivative $D^2_{X,Y}T \doteqdot D_X D_Y T - D_{\nabla_X Y} T$ can be defined invariantly. The above result holds for sections of the tensor bundle $T^r_s(E)$. In this case $\mathrm{R}_{x,y} \doteqdot D^2_{y,x} - D^2_{x,y}$. The same holds for the second Bianchi identity (namely true for general vector bundles).

For the Levi-Civita connection there is a Koszul formula, which can be checked directly. It implies the uniqueness of the Levi-Civita connection:

$$(5)\quad 2\langle Z, \nabla_Y X\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle.$$

The curvature tensor now satisfies further identities:

$$\langle \mathrm{R}_{x,y}z, w\rangle = -\langle \mathrm{R}_{x,y}w, z\rangle; \qquad \langle \mathrm{R}_{x,y}z, w\rangle = \langle \mathrm{R}_{z,w}x, y\rangle.$$

The proof of these two identities can be found almost on every book on Riemannian geometry (e.g. Do Carmo's book, page 91-92). In fact the first equation holds for a metric connection on general vector bundle. Namely $\langle \mathrm{R}_{x,y}Z, W\rangle = -\langle \mathrm{R}_{x,y}W, Z\rangle$ for Z, W sections of the vector bundle which is endowed with a metric and a metric compatible connection D. In this case R can be viewed as a operator $\mathrm{R} : \mathfrak{so}(T_pM) \to \mathfrak{so}(E_p)$.

For (M^n, g) a Riemannian manifold with $p \in M$, let γ be a closed path at p. We also use γ to denote the parallel transport along γ, which is an isometry of M_p, the tangent space at p. The Riemannian curvature tensor has a geometric interpretation via the holonomy. First for any $x, y, z, w \in M_p$, define

$$\langle \gamma(\mathrm{R}^q)_{x,y}z, w\rangle \doteqdot \langle \mathrm{R}_{\gamma^{-1}(x),\gamma^{-1}(y)}\gamma^{-1}(z), \gamma^{-1}(w)\rangle.$$

Here we also use γ to denote the parallel transport along γ from M_q to M_p.

Recall that H_p, the holonomy group at $p \in M$, is defined as the group consisting of all such $\gamma \in \mathsf{O}(M_p)$. A result of De Rham asserts that if the action of H_p on M_p is reducible then the universal cover of M splits accordingly into ΠM_i such that each factor M_i with $H_p(M_i)$ being one of the invariant subspaces. This suggests that we shall make the assumption that H_p acts irreducibly on M_p for the discussion below.

The relative holonomy group H^0_p is a path-connected subgroup of $\mathsf{O}(M_p)$, hence it is Lie subgroup of $\mathsf{SO}(M_p)$. A basic result is that H^0_p is a closed sub-group of $\mathsf{SO}(n)$ (a result of Borel-Lichnerowicz). A theorem of Ambrose-Singer (see e.g. Sternberg's Lecture for a short presentation) relates the curvature and the holonomy group.

Theorem 0.1 (Ambrose-Singer)**.** *When γ varies among all (piece-wisely smooth) pathes from q to p, and x, y vary among all vectors in M_p, $\gamma(\mathrm{R}^q)_{x,y}$ generates the holonomy algebra, namely the Lie algebra $\mathfrak{h}$, of H^0_p.*

In fact the result was formulated and proved for general principal bundles (which includes the special case of Riemannian geometry where the associated principal bundle is the orthonormal frame bundle which is a principal $\mathsf{O}(n)$ bundle). For the Riemannian holonomy group, the 1st Bianchi identity puts extra constrain on the holonomy algebra.

A fundamental result of Berger states:

Theorem 0.2 (Berger)**.** *Assume that H^0_p acts irreducibly on M_p. Then either H^0_p acts transitively on $\mathbb{S}^{n-1}$ or (M, g) is a locally symmetric space of rank ≥ 2.*

Recall that (M, g) is locally symmetric if $\nabla \mathrm{R} = 0$, namely the curvature tensor is parallel. The condition $\nabla \mathrm{R} = 0$ is equivalent to that for any p and $q \neq p$, and any path γ from q to p, $\gamma(\mathrm{R}^q) = \mathrm{R}^p$. A locally symmetric space has rank ≥ 2 if for any p, there exists $W \subset M_p$ with $\dim(W) \geq 2$ and $\mathrm{R}_{x,y} = 0$ for any $x, y \in W$. In fact by excluding subalgebras of $\mathfrak{so}(n)$ using the 1st Bianchi identity and Cartan's classification of irreducible Lie algebra, Berger gave a list of all possible holonomy groups for manifolds with are not locally symmetric: $\mathsf{SO}(n)$; $\mathsf{U}(m)$ $(n = 2m)$; $\mathsf{SU}(m)$ $(n = 2m)$; $\mathsf{Sp}(k)$ $(n = 4k)$; $\mathsf{Sp}(k) \cdot \mathsf{Sp}(1)$; Spin(9), $(n = 16)$; Spin(7), $(n = 8)$; G_2 $(n = 7)$. (The manifolds with Spin(9) holonomy were later proved to be locally symmetric.) Theorem 0.2 can then be derived from this. On the other hand using the result on the transitive actions on the sphere by Montgomery and Samuelson one can also derive the list from Theorem 0.2. If the honolomy group H^0 of (M, g) is not $\mathsf{SO}(n)$ (holonomy of generic Riemmanian manifolds) nor $\mathsf{U}(m)$ (generic Kähler manifolds) we say that (M, g) has a *special honolomy*. The above result of Berger implies that besides the locally symmetric spaces, there are only five possibilities for the special holonomy groups. The study of manifolds with special holonomy is an important subject in Riemannian geometry due to models in physics involving the geometry of the special holonomy. In particular the ones with $\mathsf{SU}(m)$ is called Calabi-Yau manifolds. For a Riemannian manifold with special holonomy group H with Lie algebra $\mathfrak{h}$, and for any loop γ at p, the parallel transport $\gamma \in H$, via part of the Ambrose-Singer Theorem (proved in the next lecture) implies that $\gamma(\mathrm{R})_{x,y} \in \mathfrak{h}$ for any x, y (recalling $R(x \wedge y) = -R_{x,y}$), in particular we have that $\gamma(R)(\mathfrak{h}) \subset \mathfrak{h}$. Lemma 0.1 says that the D is the derivative of the holonomy and holonomy is the integration of the affine connection. Hence R is the second derivative of the holonomy in some sense.

For the proof of Theorem 0.2 the following algebraic concept holds the key. We call $S = \{V, \mathrm{R}, G\}$ a *Riemannian holonomy system* if V is a Euclidean space of dimension n endowed with an inner product, G is a connected compact subgroup of $\mathsf{O}(n)$, and R is an algebraic curvature operator on V. (Namely R is a $(3, 1)$-tensor which satisfies the 1st Bianchi identity and other symmetries, abbreviated as $\mathrm{R} \in S^2_B(\wedge^2 V)$. Furthermore we assume that for any x, y, $\mathrm{R}_{x,y} \in \mathfrak{g}$, the Lie algebra of G.) The system S is called irreducible if G acts irreducibly on V. This formulation reducing the problem to an algebraic one is the first step in Simons' proof (as well as in Berger's original proof) of Theorem 0.2, which allows one to focus on algebraic issues by applying algebraic tools. This can also be seen in several works of Do Carmo-Wallach, Wallach, Wilking and the celebrated work of Böhm-Wilking on Ricci flow of manifolds with positive curvature operator (Ann. Math. 2006).

Exercises: 1. Prove Lemma 0.1. 2. Prove formula (5). 3. Prove that H^0 is normal in H.

From here the curvature R means the Riemannian curvature of the Levi-Civita connection.

4. Prove the symmetry $\mathrm{R}(x, y, z, w) = \mathrm{R}(z, w, x, y)$ for the Riemannian curvature tensor.

5. The curvature operator is defined as $\mathrm{Rm} : \wedge^2 T_p M \to \wedge^2 T_p M$ as linear extension of $\mathrm{Rm}(x \wedge y), z \wedge w\rangle = \mathrm{R}(x, y, z, w)$. Clearly Rm is a symmetric transformation. Find a R such that the sectional curvature $K > 0$ (namely $R(x, y, x, y) > 0$ for any $x \wedge y \neq 0$), but R is not positive definite.

6. Prove that if the sectional curvatures of R_1 and R_2 are the same then $\mathrm{R}_1 = \mathrm{R}_2$.

7. Prove that $3\mathrm{R}_{x,y}z = \mathrm{R}_{x+z,y}(x + z) - \mathrm{R}_{y+z,x}(y + z) + \mathrm{R}_{y,x}y + \mathrm{R}_{z,x}z - \mathrm{R}_{x,y}x - \mathrm{R}_{z,y}z$.

8. Prove that $6\mathrm{R}_{x,y}z = \mathrm{R}_{z+x,y}(z + x) - \mathrm{R}_{z-x,y}(z - x) - \mathrm{R}_{z+y,x}(z + y) + \mathrm{R}_{z-y,x}(z - y)$.

9. For a Lie algebra $\mathfrak{g}$, the Killing form $B(x, y) = \mathrm{tr}(\mathrm{ad}_x \cdot \mathrm{ad}_y)$ satisfies $B(\mathrm{ad}_z(x), y) + B(x, \mathrm{ad}_z(y)) = 0$.