We shall show that the relation between the roots puts very strong limit on the root system Δ.

This is achieved by studying, 'the α-string containing β'

$\doteq \{ \beta + n\alpha \mid n \in \mathbb{Z} \}$

$$\frac{2B(H_\beta, H_\alpha)}{B(H_\alpha, H_\alpha)}$$

(1) Theorem 4.8. Let Δ be the set of roots. Then

(a) $\exists\, p, q \geq 0$ such that the α string containing β consists of $\boxed{\text{consecutive } \cancel{\text{roots}}}$, ie $\boxed{-p \leq n \leq q}$ [No other component of $[r, s] \subset \mathbb{Z}$ integers, $\beta + n\alpha \in \Delta$, $n \in [r,s]$]

& $\boxed{\dfrac{2B(\beta, \alpha)}{B(\alpha, \alpha)} = p - q}$. $-(q-p)$, 2· the middle point

(b) If $\beta = c\alpha$ with $\alpha, \beta \in \Delta$ and $c \in \mathbb{C}$, then $c = 0$ or $c = \pm 1$.

(c) If $\alpha, \beta, \alpha+\beta \in \Delta$, then $[X_\alpha, X_\beta] \neq 0$ $\forall X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in X_\beta$.

$\boxed{n_{\alpha,\beta} := \dfrac{2B(\beta, \alpha)}{B(\alpha, \alpha)} \text{ is called a Cartan integer.}}$

Pf: Same arguement as before

Consider the $\underset{k \geq 0}{\beta + k\alpha \in \Delta}$ & $\underset{l \geq 0}{\beta - l\alpha \in \Delta}$. find the two end points

Namely $\beta + n\alpha$ $-p \leq n \leq q$ $(p, q \geq 0)$ is the longest consecutive string containing β.

Consider $V := \sum_{-p \leq n \leq q} \mathfrak{g}_{\beta + n\alpha}$

ad_{H_α} preserves V. So do ad_{X_α} & $ad_{X_{-\alpha}}$

$\Rightarrow tr(ad_{H_\alpha}) = 0 \Rightarrow \sum_n tr(ad_{H_\alpha}(\mathfrak{g}_{\beta+n\alpha})) = 0$

$$\Rightarrow \quad 0 = \sum_n (\beta + n\alpha)(H_\alpha) = (q+p+1)\, B(\alpha,\beta) + \sum_{-p \le n \le q} B(\alpha,\alpha)\, n$$

$$0 = B(\alpha,\beta) + \frac{q-p}{2} B(\alpha,\alpha)$$

$$\Rightarrow \quad \frac{2B(\alpha,\beta)}{B(\alpha,\alpha)} = q - p.$$

(b) Apply the above to α string containing β
& β string containing α.

$$\frac{2B(\alpha, c\alpha)}{B(\alpha,\alpha)} \in \mathbb{Z} \quad \& \quad \frac{2B(\frac{1}{c}\beta, \beta)}{B(\beta,\beta)} \in \mathbb{Z}$$

$$\Rightarrow \quad 2c \in \mathbb{Z} \quad \& \quad \frac{2}{c} \in \mathbb{Z} \qquad \uparrow \text{(only if } c \neq 0\text{)}$$

From this we have $c=0$, $c=\pm 1$, $c = \pm 2$, $c = \pm \frac{1}{2}$

The last two could NOT happen by the proof of Theorem 4.6.

(c) Similar argument. Assume $[X_\alpha, X_\beta] = 0$,

Consider $V = \sum_{n \le 0} \mathfrak{g}_{\beta+n\alpha}$

$\Rightarrow$ ad_{X_α} $ad_{X_{-\alpha}}$ ad_{H_α} keep it invariant.

$$\Rightarrow \quad 0 = tr(ad_{H_\alpha}|_V) = \sum_{n \le 0} (\beta + n\alpha)(H_\alpha)$$

$$= B(\alpha,\beta)(1+p) + \sum_{-p \le n \le 0} B(\alpha,\alpha)\, n$$

$$\Rightarrow \quad B(\alpha,\beta)(1+p) + (1+p) \cdot \left(\frac{-p}{2}\right) B(\alpha,\alpha) = 0$$

$$\Rightarrow \quad \frac{2B(\alpha,\beta)}{B(\alpha,\alpha)} = p$$

$\Rightarrow q=0$. Namely the string of α containing β is

$\beta+n\alpha$ for $-p\le n\le 0$. But $\beta+\alpha\in\Delta$.

[proof of (i) shows that $\nexists$ any other consecutive strings]

This is a contradiction!

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(2) The list of complex simple algebras.

A_n B_n C_n D_n — infinite series. They are called classical.

E_6 E_7 E_8, F_4, G_2 — there are called exceptional ones. The existences are NOT obvious.

Lectures on Exceptional Lie Groups. J. F. Adams.

One can get the list by studying the structure of the roots

We shall look into the classical examples first to get some feel of it.

What are the compact Lie groups?

$sl(n+1,\mathbb{C})$ A_n $\longleftrightarrow$ $SU(n+1)$, $n\ge 1$

$o(2n+1,\mathbb{C})$ B_n $\longleftrightarrow$ $SO(2n+1)$ $n\ge 2$ since $SO(3)$ covered by $SU(2)$

$o(2n,\mathbb{C})$ D_n $\longleftrightarrow$ $SO(2n)$ $n\ge 4$ — since the roots structure are different

$n=1$ is the boring case of 1-dim Lie group.
$n=2$ $SO(4)$ Not simple covered by $Sp(1)\times Sp(1)$. $SO(6)$ covered by $SU(4)$

NOT simply-connected. Its universal cover

$Spin(k) \leftarrow$ It is constructed via Clifford algebra

Ise-Takeuchi p36-39.

$C_n \longleftrightarrow Sp(n)$ — the isometries preserves the product (- $\mathbb{H}$- Hermitian) on $\mathbb{H}^n$. Page 26-27 of Ziller
$n\ge 3$
n-dimensional quaternions.

Note: $Sp(n)$ here is the same as Ziller

$Sp(n) \neq Sp(n,\mathbb{R})$, nor $sp(n,\mathbb{C})$

$Sp(n) \doteq U(2n) \cap GL(n,\mathbb{H})$. $\quad$ H – in honor of Hamilton.

We shall do A_n. $\quad \mathfrak{g} = sl(n+1,\mathbb{C})$ — the complexification of Lie $(SU(n+1))$.

$$T = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_{n+1} \end{bmatrix} \qquad \prod_{i=1}^{n+1} z_i = 1 \qquad |z_i| = 1$$

$$= \begin{bmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_{n+1}} \end{bmatrix} \qquad \sum_{i=1}^{n+1} \theta_i = 0$$

$$\Rightarrow \quad \mathfrak{t} = \begin{bmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{bmatrix} \qquad \sum_{i=1}^{n+1} \theta_i = 0$$

$$\eta = \mathfrak{t} \otimes \mathbb{C} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{bmatrix} \qquad \sum_{i=1}^{n+1} \lambda_i = 0$$

$$H = \sum \lambda_i E_{ii}$$

$\{E_{ij}\}_{i\neq j}$ satisfies

$$\begin{aligned} \mathrm{ad}_H E_{ij} &= \sum \lambda_k [E_{kk}, E_{ij}] \\ &= \sum \lambda_k (\delta_{ki} E_{kj} - \delta_{jk} E_{ik}) \\ &= \lambda_i E_{ij} - \lambda_j E_{ij} = \underbrace{(\lambda_i - \lambda_j)} E_{ij} \end{aligned}$$

Hence $\alpha_{ij}(H) = \lambda_i(H) - \lambda_j(H)$ are the roots.

The generators can be chosen as $(\underbrace{\lambda_1-\lambda_2}_{\alpha_1}, \dots \lambda_{n-1}-\lambda_n, \underbrace{\lambda_n-\lambda_{n+1}}_{\alpha_n})$

$$\alpha_{ij} = \sum_{k=i}^{j-1} \lambda_k - \lambda_{k+1} = \sum_{k=i}^{j-1} \alpha_k \qquad \underline{\dim(\eta) = n}.$$

③ $\eta_{\mathbb{R}} \doteq \sum_{\alpha\in\Delta} \mathbb{R}\cdot H_\alpha$

(This part constructs a compact Lie algebra for any real semi-simple one.)

Proposition 4.10: ⓐ $B|_{\eta_{\mathbb{R}}} > 0$

ⓑ $\eta_{\mathbb{R}}$ is the real form of η. (i.e. $\eta = \eta_{\mathbb{R}} \oplus i\,\eta_{\mathbb{R}}$)

$\mathfrak{g} = \eta \oplus \sum \mathfrak{g}_\alpha$

pf: $B(H, H') = \operatorname{tr}(\operatorname{ad}_H \operatorname{ad}_{H'})$ $X_\alpha \in \mathfrak{g}_\alpha$

$\in \eta$

$= \sum_{\alpha\in\Delta} \langle \operatorname{ad}_H \operatorname{ad}_{H'}(X_\alpha)\ \ X_\alpha^* \rangle$ (the rest $=0$ since $\operatorname{ad}_{H'}(H'')=0$ $\forall H'' \in \eta$.)

$\mathbb{C}\{X_\alpha\} \simeq \mathfrak{g}_\alpha$ is a dim 1- eigenspace.

$\alpha(H')X_\alpha$

(*)

$= \sum_{\alpha\in\Delta} \alpha(H)\alpha(H')$

Hence

$$B(H_\alpha, H_\alpha) = \sum_{\gamma\in\Delta} \gamma(H_\alpha)^2 = \sum_\gamma B(H_\gamma, H_\alpha)^2 = B(H_\alpha, H_\alpha)^2 \sum_{\gamma\in\Delta}\left(\frac{n_{\alpha\gamma}}{2}\right)^2$$

$\gamma(H) = B(H_\gamma, H)$

But $\boxed{n_{\alpha\gamma} = \dfrac{2B(H_\alpha, H_\gamma)}{B(H_\alpha, H_\alpha)}}$

$\Rightarrow B(H_\gamma, H_\alpha) = \dfrac{n_{\alpha\gamma}}{2} B(H_\alpha, H_\alpha)$

$\Rightarrow B(H_\alpha, H_\alpha) = \dfrac{4}{\sum_{\gamma\in\Delta} n_{\alpha\gamma}^2} > 0$

(**)

Hence $B(H_\beta, H_\alpha) = \dfrac{n_{\alpha\beta}}{2}\left(\dfrac{4}{\sum_{\gamma\in\Delta} n_{\alpha\gamma}^2}\right)$. real

Namely $B|_{\eta_{\mathbb{R}}}$ is real.

$\forall H \in \eta_{\mathbb{R}}$. $\exists\ \alpha\in\Delta$ such that $\alpha(H)\neq 0$ (Since $\alpha\in\Delta$ spans η^*)

$\alpha(H)\in\mathbb{R}$ by the above. Moreover $\gamma(H)$ is real

$= B(H_\alpha, H)$ $\qquad \gamma^2(H)\geq 0$

$\Rightarrow B(H.H) = \underbrace{\sum_{\gamma\in\Delta}\gamma^2(H)}_{}\geq\alpha^2(H)>0$.

By $(*)$

For (b): $\underbrace{\eta_{\mathbb{R}}\otimes\mathbb{C} \text{ clearly } \simeq \eta}$ Since $\{H_\alpha\}$ spans η.

On the other hand $\eta_{\mathbb{R}}\cap i\,\eta_{\mathbb{R}} = \{0\}$

Since $B(X,X)>0$ for X in $\eta_{\mathbb{R}}$ $X\neq 0$

$B(iX, iX) = -B(X,X)<0$ for X in $\eta_{\mathbb{R}}$.

η_R complexifies into η

η_R is the real form of η

Corollary: (a) If $\alpha, \beta\in\Delta$ $\quad \beta - \dfrac{2B(\alpha,\beta)}{B(\alpha,\alpha)}\alpha \in\Delta$ $\quad -p\leq q-p\leq q$

$\pm\alpha\neq\beta$ $\qquad \beta - \underbrace{(p-q)}\alpha$

(b) $n_{\alpha\beta}\cdot n_{\beta\alpha}\in\{0,1,2,3\}$.

[In Serre, the root system is defined using this property axiomatically]

Pf: (a) $\underline{n_{\alpha\beta} = p-q}$ $\quad \{\beta+n\alpha\}$ $-p\leq n\leq q$ is a consecutive string $\Rightarrow$ $\beta - (n_{\alpha\beta})\alpha \in\Delta$

$\qquad -q\leq \underbrace{(p-q)}\alpha \leq p$

$4\cos^2\varphi = \{0, 1, 2, 3\}$

$\Rightarrow \cos\varphi = \pm\frac{1}{2}\{\sqrt{r}\}$ $\quad r = \{0,1,2,3\}$

$\varphi = \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}\}$

(b) Since B is an inner product $\Rightarrow$ $n_{\alpha\beta}\cdot n_{\beta\alpha} = 4\cos^2\measuredangle(\alpha\ \beta)$

$\Rightarrow n_{\alpha\beta} = \dfrac{2|\alpha||\beta|\cos\measuredangle(H_\alpha\ H_\beta)}{|\alpha|^2}$ & integer

4 Not possible $\alpha\neq\pm\beta$

$$n_{\alpha\beta} := 2\,\frac{B(\beta,\alpha)}{B(\alpha,\alpha)} = 2\,\frac{|\beta|\,|\alpha|\cos\measuredangle(H_\alpha, H_\beta)}{|\alpha|^2} = 2\,\frac{|\beta|}{|\alpha|}\cos\measuredangle(H_\alpha, H_\beta)$$

$$n_{\beta\alpha} := \frac{2B(\alpha,\beta)}{B(\beta,\beta)} = 2\,\frac{|\alpha|\,|\beta|}{|\beta|^2}\cos\measuredangle(H_\alpha\, H_\beta)$$

$$n_{\alpha\beta}\cdot n_{\beta\alpha} = 4\cos^2\measuredangle(H_\alpha, H_\beta) < 4$$

$$\Rightarrow \quad \cos^2\varphi = \frac{1}{4}r \qquad r = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases}$$

$$\Rightarrow \quad \cos\varphi = \varepsilon\,\frac{1}{2}\sqrt{r} \qquad \varepsilon = \pm 1$$

$$\varphi = \begin{cases} \frac{\pi}{2} \quad \frac{3\pi}{2} & \text{if } r = 0 \\ \frac{\pi}{3} \quad \frac{2}{3}\pi & r = 1 \\ \frac{\pi}{4} \quad \frac{3}{4}\pi & r = 2 \\ \frac{\pi}{6}, \quad \frac{5\pi}{6} & r = 3 \end{cases}$$