① The proof of items ⓓ & ⓔ.

ⓓ means

$$\begin{tikzcd} \mathfrak{t} \arrow[r, hook] \arrow[d, "\pi"'] \arrow[dr, dashed] & \mathfrak{g} \arrow[d, "\bar{\pi}"] \\ \mathfrak{t}/W \arrow[r, "\sim"] & \mathfrak{g}/Ad \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, hook] \arrow[d, dashed] \arrow[dr, dashed] & G \arrow[d] \\ T/W \arrow[r, "\sim"] & G/Ad \end{tikzcd}$$

The part ⓘ of Part A lecture $\Longrightarrow$ $\forall X \in \mathfrak{g}$ $\exists g$ $Ad(g)(X) \in \mathfrak{t}$. ⓓ shows that $\mathfrak{t}/W$ & $\mathfrak{g}/Ad$ can be identified.

Given $X \in \mathfrak{t}$ $G(X)$ be the orbit $Ad(G)(X)$. if $Y \in Ad(G)(X) \cap \mathfrak{t}$ the goal is to show that $\exists n \in N(T)$ $Ad(n)(X) = Y$

$$Y = Ad(g)(X)$$

Let $Z(X) := \{ a \in G \mid Ad(a)(X) = X \}$. — Centralizer of X in G

Since $T \subset Z(X)$, $T \subset (Z(X))_0$. $\Big\{$ $Ad(\exp(tZ))(X) = e^{t\,ad_Z} X = X$ $\forall Z \in \mathfrak{t}$

$$Ad_{g^{-1}Tg}(X) = X \qquad (*)$$

Since $Ad(g^{-1}\exp(tZ)g)(X) = \frac{d}{ds}\Big|_{s=0} (g^{-1}\exp(tZ)g)(\exp(sX))(g^{-1}\exp(tZ)g)^{-1}$

$$= Ad_{g^{-1}}(Ad_{\exp(tZ)})\, Ad_g(X)$$

$$= Ad_{g^{-1}} Ad_{\exp(tZ)}(Y)$$

$$= Ad_{g^{-1}}(Y) = X$$

$\Longrightarrow (*)$

Namely $g^{-1}Tg \subset (Z(X))_0$ as well.

by Cartan-Weyl-Hopf $\Longrightarrow$ $\exists h \in (Z(X))_0$ $hTh^{-1} = g^{-1}Tg$

$\Longrightarrow (gh)(T)(gh)^{-1} = T$

$\Rightarrow \quad n = gh \in N(T)$

But $\quad Ad(n)(X) = Ad(g)\,\underline{Ad(h)(X)} = Ad(g)(X) = Y. \quad \square$

At the group level:

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, dashed] & G \arrow[d] \\ T/W \arrow[r, "\cong"] & G/Ad \end{tikzcd}$$

Namely if $x_1 \in T$ with $x_1 \in G(x_1) \cap T$ $\qquad G(x_1) = \{g x_1 g^{-1},\ g \in G\}$

the orbit

& if $G(x_1) \cap T$ has x_2

$x_2 = g x_1 g^{-1}$ for some g

We want to prove $\exists\, n \in N(T)$ such that $x_2 = n x_1 n^{-1}$

Consider $\quad Z(x_1) = Z\left(\left(\overline{\{x_1\}}\right)_0\right)$

the group generated by x_1 (a torus)

Clearly $\quad T \subset (Z(x_1))_0 \quad$ Since $a x_1 = x_1 a \quad \forall a \in T$

$g^{-1} T g \subset (Z(x_1))_0$ as well since

$$(g^{-1} a g)(x_1) = g^{-1}\,\underbrace{a(x_2)}\,g = g^{-1} x_2 a g = (g^{-1} x_2 g)(g^{-1} a g)$$
$$= x_1\, g^{-1} a g$$

Namely $g^{-1} a g$ commutes with x_1

$\Rightarrow \quad \exists\, h \in (Z(x_1))_0$ such that $h T h^{-1} = g^{-1} T g$

$\Rightarrow \quad gh T (gh)^{-1} = T \quad \Rightarrow \quad n = gh \in N(T)$

But $\quad n x_1 n^{-1} = g\,\underbrace{h x_1 h^{-1}}\,g^{-1} = g x_1 g^{-1} = x_2$

(e). Consider $Y \in t$ & $Ad(G)(Y)$, $\quad \forall\, w \in g \quad Ad(\exp(tw))(Y)$

is a path $C(t)$, $C(0) = Y$. $\quad \forall\, X \in t$

$$\langle C'(0), X \rangle = \frac{d}{dt} \langle Ad(\exp tw)(Y), X \rangle = \langle [w, Y], X \rangle = \langle w, \underbrace{[Y, X]}_{=0} \rangle$$

2 Examples

$U(n)$ — $\mathfrak{u}(n) \xrightarrow{\text{Complexified}} \mathfrak{gl}(n)$

$$B(X, Y) = \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y)$$

$X = k\,\mathrm{id} \Rightarrow$

$\ker(B) \neq 0.$

(i) $U(n)$ is NOT Semi-simple

(ii) $T = \left\{ \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, \quad |z_i| = 1 \right\}$ is a maximal torus

Pf

Since if $A \in U(n)$ $\quad AX = XA \quad \forall X \in T$

Pick a generic $X = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}$ $\quad z_i \neq z_j$

$$\Rightarrow \begin{bmatrix} z_1 a_{11} & z_2 a_{12} & \cdots & z_n a_{1n} \\ \vdots & \vdots & & \vdots \\ z_1 a_{n1} & z_n a_{n2} & \cdots & z_n a_{nn} \end{bmatrix} = \begin{bmatrix} z_1 a_{11} & \cdots & z_1 a_{1n} \\ & & \\ z_n a_{n1} & \cdots & a_n a_{nn} \end{bmatrix}$$

$\Rightarrow a_{ij} = 0$ if $i \neq j$ $\Rightarrow A \in T$

Hence T is a maximal torus

(iii) $\dim(\mathfrak{Z}(\mathfrak{g})) = 1$ with $\mathfrak{g} = \mathfrak{u}(n)$. Since $\mathfrak{su}(n)$ Complexified into $\mathfrak{sl}(n, \mathbb{C})$

$\mathfrak{sl}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C})$ is an ideal.

$B(X, Y)_{\mathfrak{sl}(n, \mathbb{C})} = 2n \operatorname{tr}(XY)$ $\quad$ pick $Y = \bar{X}^t = X^* \in \mathfrak{sl}(n, \mathbb{C})$ if $X \in \mathfrak{sl}(n, \mathbb{C})$

$\neq 0$ $\quad$ (could not $= 0 \ \forall Y$)

$\Rightarrow SU(n)$ is Semi-simple.

$\mathfrak{su}(n)$ is co-dimensional 1 in $\mathfrak{u}(n)$.

(iv) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$

$\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \to \begin{bmatrix} z_1 & & & & & & \\ & \ddots & & & & & \\ & & z_j & & & & \\ & & & \ddots & & & \\ & & & & z_i & & \\ & & & & & \ddots & \\ & & & & & & z_n \end{bmatrix}$ $\quad z_j \leftrightarrow z_i$

is obtained by Ad action.

$\Rightarrow \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \to \begin{bmatrix} z_{i_1} & & \\ & \ddots & \\ & & z_{i_n} \end{bmatrix}$ can be obtained by $Ad(g)$

(Action of W as well).

On the other hand if $g^{-1} T g \subset T$

$\Rightarrow \quad T g(e_j) = g T(e_j) = \underbrace{z_j(T)}_{\overset{\shortparallel}{e^{2\pi i \theta_j(T)}}} g(e_j)$

$\Rightarrow \quad g(e_j)$ is the common eigenvector of T

$\Rightarrow \quad g(e_j) = \lambda_j e_{\ell_j}$ $\qquad (e_1 \cdots e_n) \to (e_{\ell_1} \cdots e_{\ell_n})$ is a permutation

$\Rightarrow \quad (g^{-1} T g)(e_j) = g^{-1}(\lambda_j \cdot z_{\ell_j}(T) e_{\ell_j})$

$= z_{\ell_j}(T)\, e_j$

Namely $\quad g^{-1}\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} g = \begin{pmatrix} z_{\ell_1} & & \\ & \ddots & \\ & & z_{\ell_n} \end{pmatrix}$

In another words if $g \in N(T) \Rightarrow$ The action is inside S_n

Hence $W = S_n$.

③ root system

G — Compact Lie group.

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

$\mathfrak{z}(\mathfrak{g}) \cong \mathrm{Lie}(Z_0)$, $Z_0 \subset G$; $\mathfrak{g}_k = \mathrm{Lie}(G_k)$, $G_k \subset G$

Then $Z_0 \times G_1 \times \cdots \times G_k \longrightarrow G$

$(g_0, g_1, \cdots g_k) \longrightarrow g_0 g_1 \cdots g_k$

is a finite covering.

Hence we reduces the study to the classification of simple-compact Lie groups (up to a finite cover).

For $\mathfrak{g}$, a compact simple Lie algebra. $B < 0$ on $\mathfrak{g}$.

we reduce the study to the so-called root system.

<u>Defn</u>: Let $\mathfrak{g}$ be the complexification of a semi-simple compact Lie algebra $\mathfrak{k}$, $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$.

Let $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ the complexification of a maximum Abelian subalgebra.

Then for $\alpha \in \mathfrak{h}^*$ a linear function

$$\mathfrak{g}_\alpha := \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H) X, \ \forall H \in \mathfrak{h} \}$$ is

called the root space of α.

If $\mathfrak{g}_\alpha \neq \{0\}$ then α is called a root of $\mathfrak{g}$ (w.r.t $\mathfrak{h}$).

Let $\Delta \subset \mathfrak{h}^*$ be the set of all nonzero roots of $\mathfrak{g}$.

Δ is called the root system.

Since skew symmetric matrix can be diagonalized, the existence is easy.

General construction can be founded in Serre. [Weyl's trick reduce the general study to compact case]