1 Cartan's Theorem: $\mathfrak{g}$ a Lie algebra over $\mathbb{C}$ or $\mathbb{R}$

$B(X,Y) := \operatorname{trace}(\operatorname{ad}_X \operatorname{ad}_Y)$ the Killing form

Then $\mathfrak{g}$ is semi-simple iff B is nondegenerate.

Defn. $\mathfrak{g}$ is semi-simple if $\operatorname{Rad}(\mathfrak{g}) = \{0\}$,

$\operatorname{Rad}(\mathfrak{g}) :=$ the maximum solvable ideal of $\mathfrak{g}$.

Kind of motivated by the semi-simple, nilpotent decomposition of matrices in linear algebra [Hirsh-Smale, Appendix].

[Levi-Malcev]: $\mathfrak{g} = \eta + \operatorname{Rad}(\mathfrak{g})$, η is a Lie sub-algebra semi-simple.

$\eta \cap \operatorname{Rad}(\mathfrak{g}) = \{0\}$.

Defn: B is called non-degenerate if $\operatorname{Rad}(B) = \{ X \in \mathfrak{g} \mid B(X,Y) = 0,\ \forall Y \in \mathfrak{g} \} = \{0\}$.

[standard in linear algebra].

Reformulation: $\{0\} = \operatorname{Rad}(\mathfrak{g})$ iff $\operatorname{Rad}(B) = \{0\}$.

Lemma. $\operatorname{Rad}(B)$ is an ideal.

$x \in \operatorname{Rad}(B)$ $\quad [x, y]$ $\quad y \in \mathfrak{g}$

$\Rightarrow \quad B([x,y], z) = -B(\operatorname{ad}_y(x), z) = B(x, \operatorname{ad}_y(z)) = 0$

$\forall z \in \mathfrak{g} \quad \Rightarrow \quad [x,y] \in \operatorname{Rad}(B)$.

For proper understanding of $Rad(\mathfrak{g})$, we recall that

$\mathfrak{g}_i$ are all ideals $\quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]. \quad \mathfrak{g}_\ell = [\mathfrak{g}_{\ell-1}, \mathfrak{g}_{\ell-1}]$

The maximum solvable ideal exists & unique!

Lemma: (i) $\mathfrak{g}$ is solvable $\quad \phi: \mathfrak{g} \to \mathfrak{g}'$

$\phi(\mathfrak{g})$ is solvable

(ii) $I \subset \mathfrak{g}$ an ideal which is solvable $\quad \mathfrak{g}/I$ is solvable

$\Rightarrow$ $\mathfrak{g}$ is solvable.

(iii) $I, J \in \mathfrak{g}$ is solvable ideals $\Rightarrow$ $I + J$ is also solvable.

Pf: (i) $\phi(\mathfrak{g}_1) = [\phi(\mathfrak{g}), \phi(\mathfrak{g})] = (\phi(\mathfrak{g}))_1$

$\phi(\mathfrak{g}_2) = [\phi(\mathfrak{g}_1), \phi(\mathfrak{g}_1)] = [(\phi(\mathfrak{g}))_1, (\phi(\mathfrak{g}))_1] = (\phi(\mathfrak{g}))_2$

$\Rightarrow \mathfrak{g}_k = 0 \Rightarrow (\phi(\mathfrak{g}))_k = 0$

(ii) $\pi: \mathfrak{g} \to \mathfrak{g}/I$

$\Rightarrow$

$\pi(\mathfrak{g}_k) = (\pi(\mathfrak{g}))_k = 0$

$\Rightarrow \mathfrak{g}_k \subset I \quad \Rightarrow \left((I)_\ell = 0 \Rightarrow \mathfrak{g}_{k+\ell} = 0 \right)$

(iii) $I + J / J = I / I \cap J$

Since $\phi: I \longrightarrow I + J / J \qquad i \to \{i\}$

$\ker \phi = i$, with $\{i\} = 0 \Rightarrow i \in J \Rightarrow \ker \phi = I \cap J$

$$\Rightarrow \quad I/_{I\cap J} \cong I+J/_J$$ by linear algebra.

Hence I solvable $\Rightarrow$ $I/I\cap J$ solvable

$\Rightarrow$ $I+J$ solvable (Since J is solvable). $\square$

Hence $Rad(\mathfrak{g})$ make sense.

$I_1 = Rad(\mathfrak{g})$ — a maximum solvable ideal

$I_2 = Rad(\mathfrak{g})$ — another maximum solvable ideal

$\Rightarrow$ $I_1 + I_2$ is bigger

$\Rightarrow$ $I_1 = I_2$

$\mathfrak{z}(\mathfrak{g})$ - Center of $\mathfrak{g}$, which is Abelian $\Rightarrow \left\{ \begin{array}{l} \mathfrak{z}(\mathfrak{g}) \neq \{0\} \\ \Longrightarrow Rad(\mathfrak{g}) \neq \{0\} \end{array} \right\}$

One direction is easy:

Assume $Rad_B = \{0\}$. We will show $Rad(\mathfrak{g}) = \{0\}$.

If NoT, $\exists$ $c \subset Rad(\mathfrak{g})$ c is Abelian & $c \neq 0$.

But $\forall$ $x \in c$ $y \in \mathfrak{g}$, consider

$$ad_x \, ad_y : \underset{\in \mathfrak{g}}{z} \to [x, \underbrace{[y, z]}_{w}] \in c$$

$$\Rightarrow \left[ad_x \, ad_y\right]^2 (z) = [x, [y \underset{\in c}{w}]] = 0$$

Hence $ad_x \cdot ad_y$ is nilpotent

Linear algebra $\Rightarrow$ $trace(ad_x \, ad_y) = 0$.

$\Rightarrow B(x, y) = 0 \Rightarrow x \in Rad(B)$

But $Rad(B)=0 \Rightarrow x=0$. Namely $c=\{0\}$.

Hence $Rad(B)=0 \Rightarrow Rad(g)=0$.

2 Cartan's Criterion-

$ad: g \longrightarrow gl(g)$

$x \longrightarrow ad_x$

$ad(g) \subset gl(g)$ is an Lie sub-algebra

Fact: $\tilde{g} \subset gl(V)$ $x \in \tilde{g}$ is nilpotent iff $x^k=0$.

$\Rightarrow$ ad_x is nilpotent, as a matrix action on $gl(V)$.

$ad_x(Y)= xY-Yx = l_x - R_x$

Engel: if $\forall x \in g$ $ad_x \in gl(V)$ $V=g$

is nilpotent $\Rightarrow$ g is nilpotent

Namely identify the two concept. ($A^k=0$, $A \in M_{n\times n}$ with nilpotent Lie algebra)

Corollary: If $g \subset gl(V)$ is a sub-algebra.

$\forall x \in g$ $x^k=0$ for some k

then $\exists \{V_i\}$ - a flag, $V_i \subset V_{i+1}$ ... $V_n = V$

$\dim(V_i)=i$,

Such that $x(V_i) \subset V_{i-1}$.

Namely g is in the example we gave (in last lecture).

Cartan's Criterion: g is solvable iff B on $g^1=[g\ g]$

is zero.

One direction is via Lie's theorem:

$\tilde{g}$ is a Lie sub-algebra of $gl(V)$. if $\tilde{g}$ is solvable $\Rightarrow \exists \{V_i\}$ such that $\tilde{g}$ stablize the flag. $V_i = \text{Span}\{e_1, \dots e_i\}$

$x(V_i) = V_i \Rightarrow \tilde{g}$ is in the upper triangle Lie algebra

In particular $[\tilde{g}, \tilde{g}]$ is nilpotent. (in both senses)

Hence, if g is solvable $\Rightarrow g' = [g, g]$ is nilpotent.

$ad: g \to gl(V)$

$$ad(g') = [ad(g), ad(g)] = [\tilde{g}, \tilde{g}] = \tilde{g}'$$

Consists of nilpotent elements

$\Rightarrow g'$ is nilpotent by Engel's theorem.

$\text{trace}(ad_x \, ad_y) = 0$ clearly if ad_y ad_x are both strictly upper triangular. (By Engel's theorem apply to $ad(g')$.)

③ Proof of Cartan's theorem, — the other half.

$\mathrm{Rad}(B) \subset \mathfrak{g}$ is an ideal.

Lemma: B_h is the B restricted to h if h is an ideal.

Pf: $\{e_1 \cdots e_k\}$ basis of $\mathfrak{h}$. extend it into a basis of $\mathfrak{g}$

$$\mathrm{tr}(\mathrm{ad}_x\, \mathrm{ad}_y) = \sum_{i=1}^{n} \langle [x, \underbrace{[y, e_i]}], e_i^* \rangle$$

$x, y \in \mathfrak{h}$

$$= \sum_{j=1}^{k} \langle [x, [y, e_j]], e_j^* \rangle$$

$$+ \sum_{\ell=k+1}^{n} \langle \underbrace{[x, \underbrace{[y, e_\ell]}_{\sum_{j=1}^{k} a_j e_j}]}_{\text{Similar}}, e_\ell^* \rangle \quad 0$$

Since clearly $B|_{\mathrm{Rad}(B)} = 0 \Longrightarrow \mathrm{Rad}(B)$ is solvable.

Hence if $\mathfrak{g}$ is semi-simple $\Longrightarrow \mathrm{Rad}(B) = \{0\}$.

We did not proof Cartan's criterion (only proved the easy half).

[Lemma of 4.3 of Humphreys is the key]

Is this the hole Cartan fixed for Killing?