Two very good books. Serre: Semi-simple complex Lie algebra
Humphreys: Intr to Lie algs & representation theory.

The 2nd explains the 1st. Occasionally make it worse, so use 1st also.

(1)

Nilpotent, Solvable, Semi-simple Lie algebras

$\mathfrak{g}^k$ is defined as $\mathfrak{g}^0 = \mathfrak{g}$ $\quad \mathfrak{g}^l = [\mathfrak{g}, \mathfrak{g}^{l-1}]$ $\quad l \geq 1$

$[a, b] = \text{span}\{ [u, v], \ u \in a, \ v \in b \}$ $\quad a, b$ two ideals.

Nilpotent (of step k) if $\mathfrak{g}^k = 0$, $\mathfrak{g}^{k-1} \neq 0$.

E.g. $\mathfrak{g}_{(s)} = \{ A \in M_{n \times n} \mid a_{ij} = 0, \ \forall j \leq i+s \}$

$n-1 \geq s \geq 0$

$$\begin{bmatrix} \overbrace{0 \cdots 0}^{s+1} & a_{1,s+2} & \cdots & & a_{1n} \\ 0 \cdots 0 & 0 & a_{2,s+3} & & \\ & & & & a_{n-s-1\, n} \\ & & & & 0 \\ 0 \cdots & & & & \vdots \\ & & & & 0 \end{bmatrix} \Big\}\, s+1$$

$\mathfrak{g}_{(0)} \supset \cdots \supset \mathfrak{g}_{(n-1)}$

observe, $(x \cdot y)_{ij} = x_{ik} y_{kj} = 0$

if $j \leq i+s+1$, $x \in \mathfrak{g}_{(0)}$, $y \in \mathfrak{g}_{(s)}$

& $k \geq i+1$, $j \geq k+s+1$ $\Rightarrow$ $\boxed{j \geq i+s+2}$ is needed to have $x_{ik} y_{kj} \neq 0$

$\Rightarrow (x \cdot y)_{ij} = 0 \quad \forall \ j \leq i+s+1$

$\Rightarrow x \cdot y \in \mathfrak{g}_{(s+1)}$

Similarly, $(y\cdot x)_{ij} = y_{ik}\, x_{kj}$

$\Rightarrow$ $k \geq i+s+1$, $j \geq k+1$ $\Rightarrow$ $j \geq i+s+2$ is necessary

to have $(y.x)_{ij} \neq 0$.

This shows $\underbrace{[\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}]}$ $\subset \mathfrak{g}_{(1)}$

$[\mathfrak{g}_{(0)}, \mathfrak{g}_{(s)}] \subset \mathfrak{g}_{(s+1)}$

Hence $\mathfrak{g}_{(0)}$ is a step $n-1$ nilpotent Lie algebra.

$$a = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad n-2-\text{step} \neq 0.$$

$ad_a^{n-2}\, a \neq 0$

It is easy to see $\mathfrak{g}_{(s)}$ is $n-s-1$ step nilpotent Lie algebra.

Engel's theorem. $\boxed{\text{The general nilpotent Lie algebra is Similar to the above example.}}$

Precisely if $\forall\, \underbrace{x \in \mathfrak{g}}$ $\underbrace{(ad_x)}^k = 0$. for some $k \Rightarrow$ $\mathfrak{g}$ is nilpotent.

Pf. See page 13 of Hum.

Solvable Lie algebra: $\mathfrak{g}_k$ is defined as

$\underbrace{\mathfrak{g}_0 = \mathfrak{g}}$ $\qquad \underbrace{\mathfrak{g}_l := [\mathfrak{g}_{l-1}, \mathfrak{g}_{l-1}]}$ $\qquad l \geq 1$

k-Step solvable, if $\mathfrak{g}_k = 0$. $\mathfrak{g}_{k-1} \neq 0$

Clearly $\mathfrak{g}_l \subset \mathfrak{g}^l \Rightarrow$ Nilpotent must be solvable.

e.g.: $\mathfrak{g}_u := \{ A \mid a_{ij} = 0 \quad \forall i > j \}$

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{pmatrix} \qquad y = \text{similar}$$

$$x \cdot y = \begin{pmatrix} \underline{x_{11} y_{11}} & \cancel{*} & & * \\ 0 & \underline{x_{22} y_{22}} & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \underline{x_{nn} y_{nn}} \end{pmatrix}$$

$y \cdot x$ same form $\mathfrak{g}_u$ ~~$\mathfrak{g}$~~

$\Rightarrow \quad \underbrace{[\mathfrak{g}_u, \mathfrak{g}_u]}_{} \subset \mathfrak{g}_{(0)}$ in the last example

Now $\quad \underbrace{(\mathfrak{g}_u)_1}_{} \subset \mathfrak{g}_{(0)}$

$\Rightarrow \quad (\mathfrak{g}_u)_2 \subset \mathfrak{g}_{(1)}$

$(\mathfrak{g}_u)_3 \subset \mathfrak{g}_{(2)}$ (In fact in $\mathfrak{g}_{(3)}$)

$\Rightarrow \mathfrak{g}_u$ is solvable

<u>Lie's theorem</u>: $\forall \rho : \mathfrak{g} \to \mathfrak{gl}(N, \mathbb{C})$ then $\rho(\mathfrak{g})$ is like the example above. ($\mathfrak{g}$ is solvable)

Namely one may put $\rho(\mathfrak{g})$ upper-triangular.

<u>Pf</u>: See Hum P16.

Prop 3.2 of Ziller lists some useful facts.

linear subspace $\mathfrak{a} \subset \mathfrak{g}$ is called an ideal if $\forall u \in \mathfrak{a}$ $\forall v \in \mathfrak{g}$ $[u,v] \in \mathfrak{a}$

solvbasic

Proposition 3.2 *Let $\mathfrak{g}$ be a Lie algebra which is k step nilpotent resp. k step solvable. The following are some basic facts:*

(a) *$\mathfrak{g}_i \subset \mathfrak{g}^i$ for all i. In particular, $\mathfrak{g}$ is solvable if it is nilpotent.*

(b) *$\mathfrak{g}^i$ and $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$.*

(c) *If $\mathfrak{g}$ is nilpotent, then $\left\{\mathfrak{g}^{k-1}\right\}$ lies in the center. If $\mathfrak{g}$ is solvable, $\{\mathfrak{g}_{k-1}\}$ is abelian.*

(d) *A subalgebra of a nilpotent (solvable) Lie algebra is nilpotent (solvable).*

(e) *If $\mathfrak{a} \subset \mathfrak{b}$ is an ideal of the Lie algebra $\mathfrak{b}$, we let $\mathfrak{a}/\mathfrak{b}$ be the quotient algebra. If $\mathfrak{a}$ is solvable (nilpotent), $\mathfrak{a}/\mathfrak{b}$ is solvable (nilpotent).*

solvexact

(f) *Let*

$$0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0$$

be an exact sequence of Lie algebras. If $\mathfrak{a}$ and $\mathfrak{c}$ are both solvable, then $\mathfrak{b}$ is solvable. In general the corresponding statement is not true for for nilpotent Lie algebras.

(g) *Let $\mathfrak{a}$, $\mathfrak{b}$ be solvable (nilpotent) ideals, then the vector sum $\mathfrak{a} + \mathfrak{b}$ is a solvable (nilpotent) ideal.*

Proof We only present the proof of some of them, since most easily follow by using the Jacobi identity and induction on i.

(b) The Jacobi identity implies that $\mathfrak{g}^i$ is an ideal in $\mathfrak{g}$, and similarly $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-1}$. To see that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$, one shows by induction on k that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-k}$.

(f) Let $\phi\colon \mathfrak{a} \to \mathfrak{b}$ and $\psi\colon \mathfrak{b} \to \mathfrak{c}$ be the Lie algebra homomorphisms in the exact sequence. Clearly, $\psi(\mathfrak{b}_k) \subset \mathfrak{c}_k$. Since $\mathfrak{c}_k = 0$ for some k, exactness implies that $\mathfrak{b}_k \subset \mathrm{Im}(\mathfrak{a}_k)$ and since $\mathfrak{a}_m = 0$ for some m, we also have $\mathfrak{b}_m = 0$.

(g) Consider the exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{a} + \mathfrak{b} \to (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \to 0.$$

Since $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$, and since $\mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ are solvable ideals, $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$ is a solvable ideal as well. Thus (f) implies that $\mathfrak{a} + \mathfrak{b}$ is a solvable ideal.

The nilpotent case follows by showing that $(\mathfrak{a} + \mathfrak{b})^k \subset \sum_i \mathfrak{a}^i \cap \mathfrak{b}^{k-i}$ via induction. □

Example 3.3 a) The set of $n \times n$ upper-triangular matrices is an n-step solvable Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, and the set of $n \times n$ upper-triangular (b). $\mathfrak{g}^i$ is easier

$\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is an ideal

Since $\forall$ $[x, y]$ $\quad z \in \mathfrak{g}$

$$[\underbrace{[x, y]}_{\in \mathfrak{g}^1}, z] \in \mathfrak{g}^1$$

Now assume $\mathfrak{g}^{i-1}$ is an ideal.

We have $[\mathfrak{g}, \mathfrak{g}^{i-1}] =: \mathfrak{g}^i$

$x \in \mathfrak{g}^{i-1}$

$y \in \mathfrak{g}$

$$[\underbrace{[x, y]}_{\in \mathfrak{g}^{i-1} \text{ by the induction}}, z] \in \mathfrak{g}^i \text{ by the definition}$$

For $\mathfrak{g}_i$ it is a little harder.

$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$ is as above.

Now assume $\mathfrak{g}_{i-1}$ is an ideal

We want to show $\mathfrak{g}_i$ is an ideal.

Clearly $\mathfrak{g}_i = [\mathfrak{g}_{i-1}, \mathfrak{g}_{i-1}]$ is an ideal in $\mathfrak{g}_{i-1}$

We show inductively show $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-k}$.

Assume holds for k. & $i \geq k+1$

$\forall\ x, y \in \mathfrak{g}_{i-1}$ $\quad z \in \mathfrak{g}_{i-(k+1)}$

$$[\underbrace{[x, y]}_{\in \mathfrak{g}_i}, z] = -[\underbrace{[y, z]}_{\substack{\in \mathfrak{g}_{i-1} \\ \text{by the induction}}}, x] - [\underbrace{[z, x]}_{\in \mathfrak{g}_{i-1}}, y]$$

$\in \mathfrak{g}_i$

Hence $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-(k+1)}$