① Statement, definitions, consequences.

Peter-Weyl: (1) Let G be a compact Lie group. $f \in C(G)$.
$\forall \varepsilon > 0\ \exists\ g \in$ the representative ring of G such that
$|f - g| < \varepsilon$.

(2). If f is a class function, namely $f(\sigma x \sigma^{-1}) = f(x)$. $\forall \sigma \in G$.
then g can be chosen as a linear combination of character functions
χ_{ϕ_i} $\{\phi_i$ being irreducible representations$\}$.

· The result holds for any compact topological group.

· The 1st proof is via integral equation, the Hilbert-theory of eigenfunctions with respect to a symmetric kernel function.

See Pontyagin: Pages 225-229.
Chevalley: §XI of ch VI (pages 203-221).

· Recall for two representations (irreducible) we have proven

$$\int_G \phi_{ik} \overline{\psi_{jl}}\, du = \begin{cases} 0 & \text{if } \phi \not\sim \psi \\ \dfrac{\delta_{ij}\,\delta_{kl}}{\dim(V)} & \text{if } \phi \sim \psi \end{cases}$$

We then can collect $\left\{ \sqrt{\dim(\phi)}\ \phi_{ik} \right\}_{\phi \in \hat{G}}$ $\hat{G}$ being the equivalence class of all irreducible prepresentations of G

& $\left\{\sqrt{\dim(\phi)}\,\phi_{ik}\right\}$ forms an subsets of orthonormal elements.

Let S be the ring generated by this collection

[We shall show the linear space generated by the collection is enough]

Then (1) $\Leftrightarrow$ S is uniformly complete.

(2) $\Leftrightarrow$ The $\{$character functions of $\phi, \phi \in \hat{G}\}$. is uniformly complete among class functions.

[S is uniformly complete if $\forall f \in C(X)$, $\exists$ $f_i \in S$ $\|f - \sum a_i f_i\|_\infty < \varepsilon$]
$\varepsilon > 0$ $\uparrow$ C^0-norm.

Our proof is taken from Warner's book exercises on page 257. The idea of the proof is essentially the same as the integral-equation technique of Pontyagin.

It is a bit more precise due to the elliptic regularity, namely if $f \in C^k(G)$ one may choose $f_j \in S$

$$f_j \in C^\infty(G) \qquad \left\| f - \sum_{j=1}^{n} a_i f_j \right\|_{C^k} < \varepsilon$$

But Pontyagin's method works for G only a compact topological group!

② The proof

(A) The elliptic/Hodge theory. (also works for forms, for Riemannian vector bundle over compact mfld).

Δ: — operator. elliptic & symmetric, (≥ 0)

$\| \ \|$ - L^2-norm $\langle \ \rangle$ associated inner product (Hermitian in the case of complex bundle)

- Symmetric $\Leftrightarrow$ $\langle \Delta u, v \rangle = \langle u, \Delta v \rangle$ $\forall u, v \in \mathcal{D}om(\Delta)$.

- Ellipticity $\Leftrightarrow$ Some compactness. $\Leftrightarrow$ Some estimate + Sobolev embedding.

Key facts: (i) $\exists \{\lambda_i\}$ eigen~~space~~values of Δ

$V_\lambda := \{ u \mid \Delta u = \lambda u \}$ is finite-dimension.

(ii) $\lambda_i \to \infty$ as $i \to \infty$.

(iii) $\exists$ G - operator. which plays role as the inverse of Δ on $\ker(\Delta)^\perp$

Satisfies $G \Delta \alpha = \Delta G \alpha = \alpha, \forall \alpha \in \ker(\Delta)^\perp$

$\ker(\Delta) = V_0$ is also finite dimensional.

For the application $\Delta = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(g^{ij} \sqrt{g} \frac{\partial}{\partial x^j} \right)$

Only acts on C^∞ - functions of G. $\underbrace{\ker(\Delta) = \{c\}}_{1\text{-dimensional}}$.

(B) L^2-completeness of the eigenfunctions $\bigoplus_{i=0}^{\infty} V_{\lambda_i}$ $\qquad \underline{\lambda_0 = 0}$

$\forall\, \alpha$, consider $\left\| \alpha - \sum_{i=1}^{n} \langle \alpha, u_i \rangle u_i \right\|^2 \qquad u_i \in V_{\lambda_i}$

We assume $\{u_1 \cdots u_k\} \in \ker(\Delta)$ forms an orthonormal basis of $\underset{\ker(\Delta)}{\overset{\|}{V_0}}$

By the Hodge theory $\qquad \alpha \cancel{\in} \sum_{i=1}^{k} \langle \alpha, u_i \rangle u_i \in \ker(\Delta)^{\perp}$

$$\Rightarrow \exists\, \beta \qquad G\beta = \alpha - \sum_{i=1}^{k} \langle \alpha, u_i \rangle u_i$$

Now we claim

$$\left\| G\left(\beta - \sum_{i=k+1}^{n} \langle \beta, u_i \rangle u_i \right) \right\|^2 = \left\| \alpha - \sum_{i=1}^{n} \langle \alpha, u_i \rangle u_i \right\|^2$$

Since $\quad G\beta = \alpha - \sum_{i=1}^{k} \langle \alpha, u_i \rangle u_i$

$$\& \quad G\left(\sum_{i=k+1}^{n} \langle \beta, u_i \rangle u_i \right) = \sum_{i=k+1}^{n} \langle \beta, u_i \rangle \frac{1}{\lambda_i} u_i$$

$$= \sum_{i=k+1}^{n} \langle \beta, G u_i \rangle u_i = \sum_{i=k+1}^{n} \langle G\beta, u_i \rangle u_i$$

$$= \sum_{i \geq k+1}^{n} \left\langle \alpha - \sum_{j=1}^{k} \langle \alpha, u_j \rangle u_j,\ u_i \right\rangle u_i$$

$$= \sum_{i \geq k+1} \langle \alpha, u_i \rangle u_i$$

Hence $\Rightarrow \quad \| \alpha - \sum_{i=1}^{n} \langle \alpha, u_i \rangle u_i \|^2 = \| G \underbrace{(\beta - \sum_{i=1}^{n} \langle \beta, u_i \rangle u_i)}_{\in V_{n+1} \oplus \cdots} \|^2$

$$\Rightarrow \quad \leq \frac{1}{\lambda_{n+1}^2} \| \beta - \sum_{i=1}^{n} \langle \beta, u_i \rangle u_i \|^2$$

$$\leq \frac{1}{\lambda_{n+1}^2} \|\beta\|^2 \longrightarrow 0 \quad \text{since} \quad \lambda_{n+1} \to \infty \quad \text{as } n \to \infty$$

Hence we have proved that $\bigoplus_{i=0}^{\infty} V_{\lambda_i}$ forms a L^2-complete space

~~Then~~ In particular $\{u_i\}$ u_i orthonormal eigenfunctions for a complete L^2-basis of $L^2(G)$.

(c) $L^2 \to L^\infty$ - trick via Sobolev embedding.

Key estimate: $\quad \| \alpha \|_\infty \leq C \| (1+\Delta)^k \alpha \|$ for some $k \gg 1$ fixed only depends on $\dim(G)$.

It means if $(1+\Delta)^k \alpha \in L^2(G) \Rightarrow \alpha \in C(G)$.

Let P_n be the projection to $\bigoplus_{i=1}^{n} \{u_i\}$

namely

$$P_n \alpha = \sum_{i=1}^{n} \langle \alpha, u_i \rangle u_i$$

$$\Delta P_n \alpha = \sum_{i=1}^{n} \langle \alpha, \lambda_i u_i \rangle u_i = \sum_{i=1}^{n} \langle \Delta \alpha, u_i \rangle u_i = P_n \Delta \alpha$$

$$\|\alpha - P_n\alpha\|_\infty \leq C \| (1+\Delta)^k (\alpha - P_n\alpha) \|$$

$$= C \| (1+\alpha)^k \alpha - P_n (1+\Delta)^k \alpha \|$$

$$= C \| \varphi_0 - P_n \varphi \| \longrightarrow 0 \quad \text{as } n \to \infty$$

Hence $\Rightarrow$ $\{u_i\}$ is uniformly complete.

(one can do another approximation to apply the above to $\alpha \in C(G)$)

(D) Eigenfunction space V_λ is an representation of G.

$L_\sigma : G \to G$ is an isometry if the metric is invariant.

$\Delta \circ L_\sigma = L_\sigma \cdot \Delta$ $\qquad L_\sigma f(x) = f(\sigma x)$

$(\Delta \circ L_\sigma)(f) = L_\sigma(\Delta f(x))$.

[Laplace operator is associated with

$D(f) := \int g^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \underbrace{\sqrt{g}\, dx^1 \cdots}_{d\mu}$

$\Rightarrow$ $\Delta \cdot L_\sigma \varphi_i = L_\sigma \Delta \varphi_i = \lambda_i L_\sigma \varphi_i$

$\Rightarrow$ $L_\sigma \varphi_i$ is also an eigenfunction if $\varphi_i \in V_\lambda$

$\Rightarrow$ $L_\sigma \varphi_i = \sum_{j=1}^{k} \varphi_j G_{ji}(\sigma)$ $\qquad k = \dim(V_\lambda)$

$G_{ki}(\sigma_1\sigma_2) = (G_{kj}(\sigma_1))(G_{ji}(\sigma_2)) \Rightarrow \sigma \to (G_{ji}(\sigma))$ is a group homomorphism.

Moreover $\quad (L_\sigma \varphi_i)(e) = \sum_k \varphi_j(e)\, G_{ji}(\sigma)$

$$\Rightarrow \quad \varphi_i(\sigma) = \sum_{j=1}^{k} a_j\, G_{ji}(\sigma) \quad \Rightarrow \quad \varphi_i \in S$$

Namely φ_i is inside the representative ring.

(2) of PW theorem is based on the following observation.

If $p(x) = \sum_{i,j=1}^{r} b_i^j\, g_j^i(x)$ with $(g_j^i(x))$ being the matrix form of an irreducible representation $\phi : G \to GL(r, V)$

& $\quad p(\sigma^{-1} x \sigma) = p(x) \Longrightarrow \quad p(x) = \lambda\, \chi_\phi(x)$

$$p(\sigma^{-1} x \sigma) = \sum b_i^j\, g_j^i(\sigma^{-1} x \sigma) = \sum b_i^j\, g_k^i(\sigma^{-1})\, g_l^k(x)\, g_j^l(\sigma)$$

$$\Longrightarrow \quad \sum b_i^j\, g_k^i(\sigma^{-1})\, g_j^l(\sigma)\, g_l^k(x) = \sum b_k^l\, g_l^k(x)$$

Since $\{g_l^k(x)\}$ is independent $\Longrightarrow$

$$b_j^i\, g_k^i(\sigma^{-1})\, g_j^l(\sigma) = b_k^l$$

$$\Longrightarrow \quad \cancel{b}\, g(\sigma^{-1})\, b\, g(\sigma) = b \quad \Longrightarrow \quad b\, g(\sigma) = g(\sigma)\, b$$

$\Rightarrow \quad b = \lambda\, id \quad \Longrightarrow$ The claimed result.

(2) of PW theorem follows from this & if $f(\sigma^{-1} x \sigma) = f(x)$

& $|f - f'| < \varepsilon \Rightarrow \quad |f - f'| = \Big| f - \underbrace{\int_G f'(\sigma^{-1} x \sigma)\, d\mu_\sigma}_{= F(x)} \Big| < \varepsilon \quad$ F(x) is now a class function