Representation of Compact Lie groups

(A) Basic concepts & the main result.

$\rho: G \longrightarrow GL(r, V)$ a Lie group homomorphism is called a representation

$d\rho: \mathfrak{g} \to \mathfrak{gl}(r. V)$ is a Lie algebra representation

If $\pi_1(G) = \{e\}$, $\underset{\text{1-1 correspondence}}{\rho \longleftrightarrow d\rho}$, hence one only need to

Study one of these two.

$d\rho: \mathfrak{g} \to \mathfrak{gl}(r. V)$ — method will be more algebraic

E.g. $Ad: G \longrightarrow GL(n. \mathfrak{g})$ is a Lie group representation

Called adjoint representation.

$ad: \mathfrak{g} \longrightarrow \mathfrak{gl}(n. \mathfrak{g})$ is also called the adjoint representation

$X \to ad_X : \mathfrak{g} \to \mathfrak{g}$ $\quad ad_X(Y) = [X, Y]$

We start with 8- definitions.

1. ρ is called faithful if $\ker\rho = \{e\}$
2. V in the definition is called a G-space $\quad \rho(g).v$ also written as $g.v$.
3. $f: V \to W$. $\quad V, W$ are two G-spaces
   is called a G-map if
   $$f(g.v) = g.f(v) \iff f \circ \rho(g)(v) = \psi(g).f(v)$$

$\rho: G \to GL(r, W)$

$\psi: G \to GL(r', V)$

(4) $U \subset V$ a subspace is called invariant if $\rho(g)(U) \subset U$ $\forall g \in G$.

(5) V is called irreducible if any invariant subspace $\forall U \subset V$ must be either $U = \{0\}$ or $U = V$.

(6) $\chi_\rho(g) := \text{trace}\,(\rho(g))$ is called the character function

(7) V is called completely reducible if $V = \oplus V_i$
$\{V_i\}$ are irreducible G-spaces

(8) $\rho_i\; G \to GL(r_i, V_i)$ $i = 1, 2$ two representation are called equivalent if $\exists f : V_1 \to V_2$ linear isomorphism & a G-map.

The main theorem: (1) Two (complex) representations ρ & ψ, $\rho : G \to \underline{GL(r_1, \mathbb{C})}$, $\psi : G \to \underline{GL(r_2, \mathbb{C})}$ are equivalent iff $\chi_\rho = \chi_\psi$.

$\begin{cases} GL(r_1, V_1) & \dim_{\mathbb{C}}(V_1) = r_1 \\ GL(r_2, V_2) & \dim_{\mathbb{C}}(V_2) = r_2 \end{cases}$

(2) A (complex) representation $\rho : G \to GL(n, V)$ is irreducible iff

$$\int_G \chi_\rho(g)\, \overline{\chi_\rho(g)}\, d\mu(g) = 1$$

$d\mu$: probability Haar measure

We shall devote the rest towards the proof of this above result, which gives a complete description of the representation in terms of the character functions.

(B) Existence of a bi-invariant n-form.

Let $X_1 \cdots X_n$ be left invariant vector fields such that

$$X_i\Big|_e = \frac{\partial}{\partial x_i} \quad \text{for say I-kind canonical coordinate near } e$$

$\Rightarrow$ $\omega^L := X_1^* \wedge \cdots \wedge X_n^*$ is a n-form which is left invariant!

Let $Y_1, \cdots Y_n$ be right invariant vector field such that

$Y_i\Big|_e = \frac{\partial}{\partial x_i}$ $\Rightarrow$ $\underbrace{Y_1^* \wedge \cdots \wedge Y_n^*}_{\omega^R}$ is a Right invariant ~~vector fie~~ n-form on G

$$\left[\text{Clearly } \left[(L_x)^*\omega^L\right]_g = \left[\omega^L\right]_g \quad (L_x)^*(\omega^L(xg)) = \omega^L(g) \right]$$

$\omega^R(e) = \omega^L(e)$.

Now we show if G is compact $\Rightarrow$ ω^L is also left invariant

$\omega^R(g) = \delta(g)\,\omega^L(g)$ since ω^R, ω^L both are n-forms & $\omega^R(g) \neq 0$ $\omega^L(g) \neq 0$ $\Rightarrow$ the quotient is a well-defined function $\neq 0$.

$\delta(e) = 1$ since $\omega^R(e) = \omega^L(e)$.

We show that $\delta(gh) = \delta(g)\,\delta(h)$, namely $\delta : G \to \mathbb{R}^*$ is a group homomorphism

Since G is compact $\Rightarrow$ $\delta(g) \equiv 1$

The proof follows from the observation

$$R_g L_x = L_x R_g \qquad \left(\begin{matrix} \forall\, \sigma \\ R_g L_x(\sigma) = \quad x\sigma g = L_x R_g(\sigma) \end{matrix} \right)$$

$$\Rightarrow \quad L_x^* R_g^* = R_g^* L_x^*$$

Apply to $\omega^L \Rightarrow \qquad R_g^* \omega^L = L_x^* \left(R_g^* \omega^L \right)$

$\Rightarrow \quad R_g^* \omega^L$ is also left invariant

$$\Rightarrow \quad R_g^* \omega^L = C(g)\, \omega^L \qquad \left(\begin{matrix} \text{Two left invariant } n\text{-forms} \\ \text{differs by a constant} \end{matrix} \right)$$

$$\Rightarrow \quad R_g^* \left(\delta^{-1}(xg)\, \omega^R(xg) \right) = C(g) \left(\delta^{-1}(x)\, \omega^R(x) \right)$$

$$\Rightarrow \qquad \delta^{-1}(xg) = C(g)\, \delta^{-1}(x)$$

$$\Rightarrow \quad \frac{1}{\delta(xg)} = \frac{\cancel{\delta(x)}}{\cancel{C(g)}} \frac{C(g)}{\delta(x)} \qquad x = e \;\Rightarrow$$

$$\frac{1}{\delta(g)} = \frac{\cancel{\delta(e)}}{C(g)} \;\Rightarrow$$

$$\frac{1}{\delta(xg)} = \frac{\cancel{\delta(g)}\; 1}{\delta(x)\,\delta(g)} \quad \Rightarrow \qquad \delta(xg) = \delta(x)\,\delta(g)$$

Namely δ is a group homomorphism.

Corollary: · $\forall$ Any real representation $\rho: G \to GL(r, \mathbb{R})$
$GL(r, V)$ $\dim_{\mathbb{R}} V = r$

$\exists$ an inner product on V which is G-invariant.

· $\forall$ Any complex representation $\rho: G \to GL(r, V)$

$\exists$ a Hermitian product on V which is G-invariant

$\langle gv, gw \rangle = \langle v, w \rangle$ $\quad$ ($\langle \ \rangle$ is g invariant $\Leftrightarrow g \in O(V)$)

For complex case $(gv, gw) = (v, w)$ $\quad$ $(\ )$ Hermitian product

$\Leftrightarrow g \in U(V)$.

Pf: Pick any $\langle\langle \ \rangle\rangle$ on V define

$$\langle v, w \rangle = \int_G \langle\langle gv, gw \rangle\rangle \, d\mu(g)$$

$$\langle \sigma v, \sigma w \rangle = \int_G \langle\langle \underbrace{\sigma g}_{g'} v, \sigma g w \rangle\rangle \, d\mu(g)$$

$$= \int_G \langle\langle g' v, g' w \rangle\rangle \, \underbrace{d\mu(\sigma^{-1} g')}_{= d\mu(g')} = \langle v, w \rangle$$

The complex case is similar.

Thm: Any real / Complex representation of G, a compact Lie group is completely reducible.

Pf: $O(V)$, $U(V)$ element $\exists$ Canonical form.

$U(V)$ can be similar to a diagonal matrix

$O(V)$ Can be put into 2x2-block diagonal form.

Each one is clearly invariant & irreducible.

(C) The proof of the main result.

Schur's Lemma: Let V, W be two irreducible G-spaces. Let $A: V \to W$ be a G-map. Then either $A \equiv 0$ or A is an isomorphism

Pf: $\ker A$ & $\mathrm{Im}(A)$ are G-invariant subspaces

E.g. $g \cdot (Av) = A(g \cdot v) \in \mathrm{Im}(A)$ $\square$

Pf: If ρ & ψ are equivalent $\Rightarrow$ $\chi_\rho(g) = \chi_\psi(g)$

is due to $\mathrm{trace}(A) = \mathrm{trace}(B^{+} A B)$

~~To do this~~ ρ & ψ are equivalent $\Leftrightarrow$ $\exists f: V \to W$

$f(g \cdot v) = g f(v)$ $\qquad$ $\{e_i\}$, $\{e_i^*\}$ base of V, V^*

$f(\rho(g) v) = \psi(g) f(v)$ $\qquad$ $\{\tilde{e}_s\}$ $\{\tilde{e}_s^*\}$ — of W, W^*

$(\rho(g))(e_i) = e_j (\rho(g))_{ji}$ $\qquad$ $f(e_i) = \tilde{e}_s f_{si}$

$$f(\rho(g).(e_i)) = f(e_j \, \rho(g)_{ji}) = \tilde{e}_s \, \underline{f_{sj} \, \rho(g)_{ji}}$$

$$\psi(g) f(e_i) = \psi(g)(e_t \, f_{ti}) = \tilde{e}_s \, \psi(g)_{st} f_{ti}$$

$$\Rightarrow \quad f \, \rho(g) = \psi(g) \cdot f \Rightarrow \quad \psi(g) = f \cdot \rho(g) f^{-1}$$

(As matrices)

$$\Rightarrow \text{trace } \psi(g) = \text{trace}(\rho(g)) \Rightarrow \quad \chi_\psi = \chi_\rho$$

To prove the other direction, we consider two irreducible representation ϕ. & ψ

$$\phi(g)(e_i) = e_j (\phi(g))_{ji} \quad (= e_j \, \phi_{ji})$$

$$\psi(g)(e_s) = e_t \, (\psi(g))_{ts} \quad (= e_t \, \psi_{ts})$$

$E_{ab} : V \to W$
$$E_{ab}(e_i) = \begin{cases} 0 & \text{if } b \neq i \\ \tilde{e}_a & \text{if } b = i \end{cases}$$

Construct the map $V \to W$

$$A_{ab} := \int_G \psi(g) \cdot E_{ab} \cdot \phi(g^{-1}) \, d\mu(g) \qquad V \to W$$

$$A_{ab}(\phi(\sigma^{-1})) = \int_G \psi(g) \cdot E_{ab} \cdot \phi(g^{-1}\sigma^{-1}) \, d\mu(g)$$

$$= \int \psi(\sigma^{-1}) \, \psi(\sigma g) \, E_{ab} \, \phi((\sigma g)^{-1}) \, d\mu(g)$$

$$= \psi(\sigma^{-1}) \, A_{ab} \qquad \forall \sigma \in G \Rightarrow A_{ab} \text{ is a } G\text{-map}$$

By Schur's Lemma $\Longrightarrow$ if $\phi \not\sim \psi$ then $A_{ab} \equiv 0$

This implies
$$\int_G \psi_{sa}(g)\, \overline{\phi_{ib}}(g) = 0 \qquad \forall \quad \begin{matrix} 1 \le i, b \le n \\ 1 \le s, a \le m \end{matrix} \qquad (*1)$$

Prove: Consider action of $A_{ab}(e_i) \Longrightarrow$

$$\int_G \psi(g) \cdot E_{ab} \cdot \underline{\phi(g^{-1})(e_i)}$$

$$\phi(g)^{-1}(e_i) = {}^t\overline{\phi(g)}\,(e_i) = e_j \overline{\phi}_{ij}$$

$$\Rightarrow \quad A_{ab}(e_i) = \int \psi(g)\, E_{ab}(e_j)\, \overline{\phi}_{ij} = \int \delta_{bj}\, \psi(g)\,(\tilde{e}_a)\, \overline{\phi}_{ij}$$

$$= \sum \tilde{e}_s \int \psi_{sa} \overline{\phi_{ij}}\, \delta_{bj}$$

$$= \sum \tilde{e}_s \int \psi_{sa} \overline{\phi_{ib}} \qquad = 0$$

$$\Rightarrow \int \psi_{sa} \cdot \overline{\phi_{ib}} = 0 \qquad (*1)$$

$$(*1) \Rightarrow \qquad \int \chi_\psi \overline{\chi}_\phi = 0$$

Case 2 $\psi = \phi \Rightarrow$

$$\int \phi_{sa} \overline{\phi_{is}} = \frac{\delta_{si}\delta_{ab}}{\dim V}$$

Hence $\int_G \chi_\phi \overline{\chi_\phi} = 1$

Now let $\hat{G}$ denotes all irreducible representation of G

$\{\chi_\rho . \rho \in \hat{G}\}$ for set of orthonormal functions of $L^2(G, du)$

$\forall \rho = \sum m(\rho, \phi_i)\phi_i$ $\qquad$ $m(\rho . \phi_i)$ are multiplicities

$\chi_\rho = \sum m(\rho . \phi_i) \chi_{\phi_i}$

$\int \chi_\rho \cdot \overline{\chi_{\phi_j}} = m(\rho\ \phi_j)$

Hence if $\chi_\rho = \chi_\psi \Rightarrow m(\rho . \phi_j) = m(\psi . \phi_j)$

$\Rightarrow \rho = \psi$

For (2) of main theorem $\Rightarrow \int \chi_\rho \overline{\chi_\rho} = 1$

$\Rightarrow \sum m^2(\rho . \phi_i) = 1 \qquad m(\rho . \phi_i) \in \mathbb{Z}^+$

$\Rightarrow \rho = \phi_{i_0}$ for some i_0 $\Rightarrow$ ρ is irreducible.