① Remarks

(A) σ-system & abstract root system

If $\Delta \subset \eta$ is the root system. We have that
from previous definition

$\forall\ \alpha, \beta \in \Delta$

$$\beta + n\alpha \in \Delta \qquad -p \le n \le q, \quad p \ge 0,\ q \ge 0$$

historically it is called a σ-system

with $$p - q = n_{\beta\alpha} = 2\,\frac{B(\beta,\alpha)}{B(\alpha,\alpha)}$$

I may used a wrong order.

In particular $$\beta - 2\,\frac{B(\beta,\alpha)}{B(\alpha,\alpha)}\,\alpha \in \Delta$$

$$\boxed{\beta - (p-q)\alpha \in \Delta \quad \text{Since} \quad -p \le n = q-p \le q}$$

The observation is if $B(\cdot,\cdot)$ is viewed as an inner product $$S_\alpha(\beta) = \beta - 2\,\frac{(\beta,\alpha)}{(\alpha,\alpha)}\,\alpha$$ is the reflection w.r.t. to $P_\alpha := \{ x \in \eta_{\mathbb{R}} \mid (x,\alpha) = 0 \}$

The abstract root system only requires Δ is invariant under S_α (the reflections), i.e (0) $\forall\ \beta \in \Delta \quad S_\alpha(\beta) \in \Delta$.

together with (i) Δ is finite, & $\mathrm{Span}\{\Delta\}$ is $\eta_{\mathbb{R}}$

& (ii) only multiple of $\alpha \in \Delta$ is $-\alpha$

(iii) $\beta - S_\alpha(\beta)$ is an integer multiple of α (Not requiring in Δ)

(B) The Weyl (abstract) group

$W(\Delta) :=$ the group generated by $\{S_\alpha\}$, the reflection above.

For the case $\mathfrak{g} = k \otimes \mathbb{C}$ $\mathfrak{h} = t \otimes \mathbb{C}$, it can be shown that $\boxed{\text{the two Weyl groups are the same.}}$

It also satisfies many nice properties:

(i) $W(\Delta)$ acts transitively on the Weyl chambers

$WC :=$ the connected components of $\mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Delta} P_\alpha$.

(ii) $\forall H_0'$ regular, $\exists\ \sigma \in W(\mathfrak{g})$

$(\sigma(H_0'), \alpha) > 0 \quad \forall\ \alpha \in F$

(iii) $\forall\ \alpha \in \Delta$, $\exists\ \sigma \in W(\Delta)$ $\sigma(\alpha) \in F$

(iv) $W(\Delta)$ is generenated by $\{S_\alpha\}$ with $\alpha \in F$.

(v) If F' is another Fundamental root-set, then $\exists\ \sigma \in W(\mathfrak{g})$

$\sigma(F') = F$.

This is mainly proved in Serre (5.10) & Humphreys (10.3).

(C) maximal root

(i) It depends the ordering choice.

(ii) $\exists$ several definitions. I will try to use the pairing with H_0 - the regular vector in $\mathfrak{h}_R$.

Def: $\beta \in \Delta^+$ is called a (H_0)-maximal root, if

(H_0) $\quad (\beta, H_0) \geq (\alpha, H_0) \quad \forall \alpha \in \Delta^+$

$(*)$-maximal if $\alpha_m + \beta \notin \Delta^+, \forall \beta \in \Delta^+$ $(*)$

Proposition 4.48: For a simple Lie algebra $\mathfrak{g}$

(a) If α_m is an element in Δ^+ with maximal height $\Longrightarrow$ α_m is $(*)$-max

Namely $ht(\alpha_m) \geq ht(\beta), \forall \beta \in \Delta^+$ (ht)

(b) $\alpha_m = \sum n_i \alpha_i$, $\quad \alpha_i \in F$, $n_i > 0$

& α_m is unique.

(c) If $\beta = \sum m_i \alpha_i \in \Delta^+$ $\quad m_i \geq 0$

$\Longrightarrow$ $\quad m_i \leq n_i$

In particular, if $\beta \neq \alpha_m$ $\quad \beta < \alpha_m$

(d) α_m is the unique (H_0) maximal root in Δ^+

Pf: $ht(\alpha) \doteq \sum n_i$ $\quad$ if $\alpha = \sum n_i \alpha_i$ $\quad \alpha_i \in F$

Since $\forall \alpha \in \Delta^+$, such an expression exists.

& it is unique due to $\{\alpha_i\}_{\alpha_i \in F}$ forms a basis of $\mathfrak{h}_{\mathbb{R}}$

We first show if α_m has the maximal height, then

α_m is maximal in the sense that it satisfies $(*)$.

(a) If α_m has maximal height. $\Longrightarrow$ $\alpha_m + \alpha_i \notin \Delta^+$

Hence $(*)$ holds if $ht(\beta) = 1$. We say $(*)$ holds.

Now we prove it by induction on $ht(\beta)$.

Assume it holds for $ht(\beta) \leq k$. Namely $(*_k)$ holds.

Lemma: $\forall\ \beta \in \Delta^+$, $ht(\beta) = k+1$, $\exists\ \alpha_j \in F$ and $\beta' \in \Delta^+$ such that $\beta = \beta' + \alpha_j$, $k \geq 1$

Pf: (i) $(\beta, \beta) > 0$.

$$\beta = \sum n_j \alpha_j \qquad \alpha_j \in F$$

$$\Rightarrow \quad \sum n_j (\alpha_j, \beta) > 0$$

Hence, $\exists\, j_0$ such that $n_{j_0} > 0$ & $(\alpha_{j_0}, \beta) > 0$

otherwise
$$\sum n_j (\alpha_j, \beta) \leq 0$$

Pick such j_0: $(\beta, \alpha_{j_0}) > 0 \Rightarrow \beta - \alpha_{j_0} \in \Delta$

But
$$\beta = \sum n_j \alpha_j = \sum_{j \neq j_0} n_j \alpha_j + (n_{j_0} - 1)\alpha_{j_0} + \alpha_{j_0}$$

$$\beta' = \underbrace{\sum_{j \neq j_0} n_j \alpha_j + (n_{j_0} - 1)\alpha_{j_0}}_{= \beta - \alpha_{j_0}} \in \Delta \quad \& \quad \in \Delta^+ \qquad \square$$

Write $\beta = \beta' + \alpha$ with $ht(\beta') = k$ & $\alpha \in F$ via the Lemma.

Recall $\underset{\alpha\in F}{\underline{\quad}}$ $X_\alpha \in \mathfrak{g}_\alpha$, $X_{\beta'} \in \mathfrak{g}_{\beta'}$ Theorem 4.8 (c)

$\alpha+\beta'=\beta\neq 0 \Longrightarrow [X_\alpha, X_{\beta'}]\neq 0 \in \mathfrak{g}_\beta$

Moreover if $\underbrace{\alpha_m+\beta}\in\Delta^+ \Longrightarrow$

$$[X_m, [X_\alpha, X_{\beta'}]]\neq 0$$

But $\quad \parallel$

$$-[X_\alpha, [X_{\beta'}, X_m]] - [X_{\beta'}, [X_m, X_\alpha]]$$

Since $\alpha_m+\beta'\notin\Delta^+ \Rightarrow [X_{\beta'},X_m]=0$ & $\alpha_m+\alpha\notin\Delta^+ \Rightarrow [X_m,X_\alpha]=0$

$(ht(\beta')=k$ $\quad (\notin\Delta\Rightarrow\notin\Delta^+)$ $\quad$ $ht=1$ case

This is a contradiction.

Hence for α_m which has maximal $ht(\alpha_m)\Rightarrow (*)$ holds.

Namely $\alpha_m+\beta\notin\Delta^+$ $\forall\beta\in\Delta^+$. I.e α_m is $(*)$-maximal.

$\alpha_m=\sum n_k\alpha_k$ $\quad \alpha_k\in F$

(b) $\Delta_1 := \{\alpha_i \in F \mid n_i>0\}$ $\qquad F=\Delta_1\cup\Delta_2$

$\Delta_2 := \{\alpha_j \in F \mid n_j=0\}$

$(\alpha_m,\alpha_j)\leq 0 \quad \forall\alpha_j\in\Delta_2$,

Since $(\alpha_m, \alpha_j) = (\sum_{\alpha_i\in\Delta_1} n_i\alpha_i, \alpha_j)\leq 0$

But $(\alpha_m, \alpha_j)\geq 0$ otherwise $\alpha_m+\alpha_j\in\Delta^+$ $\Big\}$ $ht(\alpha_m+\alpha_j) = ht(\alpha_m)+1$

$((*_1)$-holds is enough) $(\star)$

$\Longrightarrow (\alpha_m, \alpha_j)=0$, But $(\alpha_i, \alpha_j)\leq 0$

$n_i>0$

$\Rightarrow \quad (\alpha_i, \alpha_j) = 0 \quad \forall \ \underbrace{\alpha_i \in \Delta_1}\ \ \& \ \underbrace{\alpha_j \in \Delta_2}$

This $\Longrightarrow$ $\underbrace{D(\mathfrak{g}) \text{ is disconnected}}$, unless $\Delta_2 = \phi$ [$\mathfrak{g}$ simple iff $D(\mathfrak{g})$ is connected]

Now we show α_m is a unique one which has $ht(\alpha_m)$ is maximum.

If β_m is another element in Δ^+ [$ht(\beta_m) = ht(\alpha_m)$].
such that β_m also has the maximal height.

If (i) $(\alpha_m \ \beta_m) > 0$ & $\underline{\alpha_m \neq \beta_m}$ $\Rightarrow$, $\alpha_m - \beta_m \in \Delta$

$\Rightarrow$ either $\alpha_m - \beta_m \in \Delta^+$ or $\beta_m - \alpha_m \in \Delta^+$

$\Rightarrow$ $\alpha_m = \beta_m + \underbrace{(\alpha_m - \beta_m)}_{\in \Delta^+}$ $\qquad \beta_m = \alpha_m + \underbrace{(\beta_m - \alpha_m)}_{\in \Delta^+}$

All contradicts to $(*)$. [Maximal in ht $\Rightarrow$ maximal in $(*)$]

(ii) if $(\alpha_m, \beta_m) = 0 \Rightarrow \sum n_j (\alpha_j, \beta_m) = 0$

But $\boxed{(\alpha_j, \beta_m) \geq 0}$ $\Longrightarrow$ $(\alpha_j, \beta_m) = 0 \Rightarrow \beta_m = 0$.

by $(*)$ or $ht(\beta_m)$ maximal $\quad$ & $n_j > 0$

If $(\alpha_j, \beta_m) < 0 \Rightarrow \beta_m + \alpha_j \in \Delta \Rightarrow \beta_m$ is Not of maximal height.

(iii) If $(\alpha_m, \beta_m) < 0 \Rightarrow \underbrace{\alpha_m + \beta_m} \in \Delta \Rightarrow \alpha_m + \beta_m \in \Delta^+$

$\Rightarrow$ neither of them is of maximal height.

The contradiction forces $\alpha_m = \beta_m$.

(c) If $\beta \neq \alpha_m \quad \beta \in \Delta^+$.

(b) implies (c) holds at $ht(\beta) = ht(\alpha_m)$

the uniqueness part

& Also if $ht(\beta) = 1$, it holds.

Now if $ht(\beta) = i$ & $\boxed{\text{ⓒ holds for } ht(\beta) > i}$ Claim: $\exists\, \alpha_i \in F$

$\beta + \alpha_i \in \Delta^+$, otherwise we have β satisfies $(*_1)$.
$ht(\beta) > 1$ by the proof of ⓐ

Hence ⓑ applies. $(\star)$ $\Rightarrow$ $\beta = \sum n_i \alpha_i$ $n_i > 0$. β satisfies $(*)$

Also $\beta = \alpha_m$ (by ⓑ arguement, $\alpha_m \leftarrow \beta$, $\beta_m \leftarrow \alpha_m$)

Hence $\beta' = \beta + \alpha_i$ has $ht(\beta') = i+1$.

$\Rightarrow$ ⓒ holds for β' (by induction assumption)

$\Rightarrow$ It also holds for β. Namely $m_i \leq n_i$.

Corollary of ⓒ is

$(\alpha_m, H_0) \geq (\beta, H_0)$ Since $(\alpha_i, H_0) > 0$.

Moreover if $\beta \neq \alpha_m \Rightarrow (\alpha_m, H_0) > (\beta, H_0)$.

Namely α_m is maximal in terms of $h_0(H) := (H, H_0)$

ⓓ Follows by the construction. & ⓐ, ⓑ, ⓒ.

2 Weight, Weight spaces, the maximal Weight.

Now we consider representation of simple Lie algebras.

$\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ linear map preserves the $[\ ,\ ]$.

We consider irreducible representations.

Defn: $\mu \in \eta^*$ is called a weight if

$$V_\mu := \{ v \in V \mid H.v = \mu(H) v, \ \forall H \in \eta \}$$

Here $H.v$ means $\varphi(H)(v)$.

The representation allows us to view V as a $\mathfrak{g}$-module.

$$(X.Y.v) = X.(Y.v) = \varphi(X)(\varphi(Y)(v))$$

The vector space V_μ is called the weight space.

If $\mathfrak{g}$ is the Lie algebra of a compact Lie group.

$\Rightarrow \exists$ metric on V such that $\tilde{\varphi}(g) \in U(V)$

$\Rightarrow$ Any one can be diagonized.

Same applies to $\varphi(X)$. $X \in \mathfrak{g}$. since $\varphi(X) \in \mathfrak{u}(n)$

Our discussions on root & root spaces works.

In particular,

Lemma: $\forall \alpha \in \Delta$. $X_\alpha \in \mathfrak{g}_\alpha$ & μ a weight

$$X_\alpha \cdot V_\mu \subset V_{\mu+\alpha}$$

Pf: $H.X_\alpha.v = X_\alpha.\underbrace{H.v}_{} + \underbrace{[H, X_\alpha]}_{}.v$

$$= \mu(H) X_\alpha.v \qquad \alpha(H) X_\alpha.v$$

$$= (\mu(H) + \alpha(H)) X_\alpha.v$$

Defn: $\mu_1 \leq \mu_2$ if $\underline{\mu_1(H_0) \leq \mu_2(H_0)}$

$\in \eta_R^*$

An order in the weight system.

In order to make sense out of it we need the following

Proposition: (a) V is the direct sum of weight spaces

(b) If μ is a weight, $\mu(T_\alpha) \in \mathbb{Z}$ $\forall$ $\alpha \in \Delta$.

$$T_\alpha = \frac{2H_\alpha}{(H_\alpha, H_\alpha)} \implies \mu(\eta_R) \text{ is real.}$$

Pf (a) By the eigen-space decomposition for a family of commutative diagonizable (Semi-simple) operators.

(b) observe that T_α X_α $X_{-\alpha}$ satisfies

$$[X_\alpha, X_{-\alpha}] = T_\alpha$$

$$[T_\alpha\ X_\alpha] = \alpha(T_\alpha) X_\alpha = 2(H_\alpha\ T_\alpha) X_\alpha = 2X_\alpha$$

$$[T_\alpha\ X_{-\alpha}] = -\alpha(T_\alpha) X_{-\alpha} = -2\ X_{-\alpha}$$

Hence span $\{T_\alpha, X_\alpha, X_{-\alpha}\}$ generates a $\mathfrak{sl}(2, \mathbb{C})$.

The representation $\varphi: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$

restrict to $\varphi|_{\mathfrak{sl}(2,\mathbb{C})} \longrightarrow \mathfrak{gl}(V)$ is a representation of

$Sl(2,\mathbb{C})$. $\qquad u(T_\alpha)$ then are all integers.

Let G be a compact Lie group. $\varphi: G \to GL(V)$ is a representation

T be a maximum torus.

Theorem: ⓘ $Q(H) \doteq \sum_{\sigma \in W} \text{Sign}(\sigma)\, e^{2\pi\sqrt{-1}\,(\sigma(\delta), H)} \neq 0$

for generic $H \in \mathfrak{t}$.

$\delta = \frac{1}{2} \sum_{\alpha_i \in \Delta^+} \alpha_i$

(ii)
$$\chi_\varphi(\text{Exp}\, H) = \frac{\sum_{\sigma \in W} \text{Sign}(\sigma)\, e^{2\pi\sqrt{-1}\,(\sigma(\Lambda_\varphi + \delta), H)}}{Q(H)} \qquad (W)$$

Λ_φ is the highest weight of φ

$\text{Exp}(H) \in T \subset G$

the maximal torus.

(iii) Two complex irreps. φ & ψ of G

irreducible representations

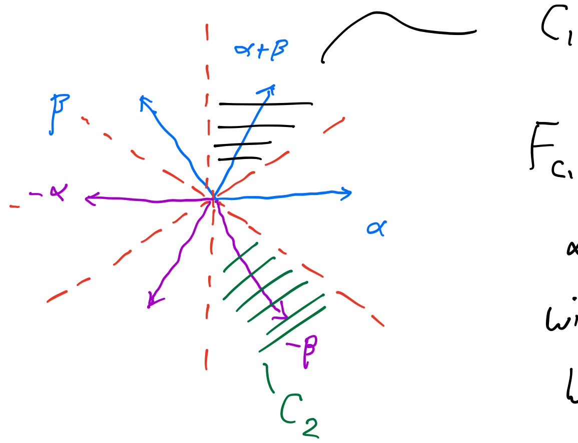
are equivalent iff $\Lambda_\varphi = \Lambda_\psi$.

(iv)
$$\dim \varphi = \prod_{\alpha \in \Delta^+} \frac{(\Lambda_\varphi + \delta, \alpha)}{(\delta, \alpha)}$$

Namely the dimension of the representation is determined by Λ_φ, the maximal / highest weight.

Question: Is there any relation between Weyl's formula (W) & Hirzebruch-Riemann-Roch?

③ E.g.



$F_{C_1} = \{\alpha, \beta\}$

$\alpha+\beta$ is the one with the highest weight

$F_{C_2} = \{\alpha, -(\alpha+\beta)\}$

$-\beta = \alpha + (-(\alpha+\beta))$ is the one with the highest weight

An important property for δ.

④ $S_\alpha : \Delta^+ \setminus \{\alpha\} \longrightarrow \Delta^+ \setminus \{\alpha\}$.

Pf. $S_\alpha(\beta) = \beta - 2\dfrac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha$

Since $\forall \beta \in \Delta^+ \setminus \{\alpha\}$ $\beta = \sum n_j \alpha_j$ Some $\alpha_{j_0} \neq \alpha$ $n_{j_0} > 0$

Otherwise β is a multiple of α $\beta \in \Delta^+ \Rightarrow \beta = \alpha$.

But $S_\alpha(\beta) = (\sum n_j' \alpha_j)$ expression still holds

& either $n_j' \geq 0$ $\forall j$ or $n_j' \leq 0$ $\forall j$

But the reflection clearly does not affect the coefficient n_{j_0} since it could only affect multiples of α.

(b) $S_{\alpha_i}(\delta) = \delta - \alpha_i$ $\quad \alpha_i \in F$

Pf: $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \frac{1}{2} \sum_{\substack{\alpha \in \Delta^+ \\ \alpha \neq \alpha_i}} \alpha + \frac{1}{2}\alpha_i$

$$S_{\alpha_i}(\delta) = \frac{1}{2} \sum_{\substack{\alpha \in \Delta^+ \\ \alpha \neq \alpha_i}} \alpha - \frac{1}{2}\alpha_i = \frac{1}{2}\delta - \alpha_i$$

(c) $\frac{2(\alpha_i, \delta)}{(\alpha_i, \alpha_i)} = 1$

Pf: $S_{\alpha_i}(\delta) = \delta - \frac{2(\delta, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i = \delta - \alpha_i$

$$\Rightarrow \frac{2(\delta, \alpha_i)}{(\alpha_i, \alpha_i)} = 1 \qquad \forall \alpha_i \in F.$$

(d) General formula:

$$(*) \quad \sum_{\sigma \in W} \operatorname{sign}\sigma \; e^{2\pi\sqrt{-1}(\sigma(\delta), H)} = \prod_{\alpha \in \Delta^+} \left(e^{\pi\sqrt{-1}(\alpha, H)} - e^{-\pi\sqrt{-1}(\alpha, H)} \right)$$

Then $\sum_{\sigma \in W} \operatorname{sign}\sigma \; e^{2\pi\sqrt{-1}(\sigma(\Lambda_\varphi + \delta), t\delta)} = \sum_{\sigma \in W} \operatorname{sign}\sigma \; e^{2\pi\sqrt{-1}(\sigma(t\delta), \Lambda_\varphi + \delta)}$

$$= \prod_{\alpha \in \Delta^+} \left(e^{\pi\sqrt{-1}(\alpha, (\Lambda_\varphi + \delta)t)} - e^{-\pi\sqrt{-1}(\alpha, (\Lambda_\varphi + \delta)t)} \right)$$

Hence:

$$\dim(\varphi) = \lim_{t\to 0} \chi_\varphi(t\delta) = \lim_{t\to 0} \frac{\prod_{\alpha\in\Delta^+}\left(e^{\sqrt{-1}\pi(\alpha,(\Lambda_\varphi+\delta)t)} - e^{-\sqrt{-1}\pi(\alpha,(\Lambda_\varphi+\delta)t)}\right)}{\prod_{\alpha\in\Delta^+}\left(e^{\sqrt{-1}\pi(\alpha,\delta t)} - e^{-\sqrt{-1}\pi(\alpha,\delta t)}\right)}$$

by the definition of χ, the character function.

$$= \prod_{\alpha\in\Delta^+} \frac{(\Lambda_\varphi+\delta,\alpha)}{(\delta,\alpha)}.$$

This proves (iv) of Theorem.

One can consult Hsiang's book for a proof of (*) and the proof of (i) & (ii) of the theorem.

(iii) is an easy consequence of the character theory we covered before & the Cartan-Weyl-Hopf theorem.