(1) Lie groups & Lie algebras (Lie's fundamental theorems 1842–1899)

Def 1.a G is called a Lie group if

(i) G is a smooth manifold & a group

(ii) $\varphi : g \longrightarrow g^{-1}$

$\psi : G \times G \longrightarrow G \ni \psi(g_1, g_2) = g_1 g_2$, (g_1, g_2) } are smooth

Remark: (i) $\varphi_1 : G_1 \longrightarrow G_2$ (smooth) $\varphi_1(g_1 g_2) = \varphi_1(g_1) \cdot \varphi_1(g_2)$ is called a homomorphism between Lie groups.

Any homomorphism is analytic

(ii) Any Lie group is analytic [In fact a topological group with G being a topological manifold is a Lie group]

Eg. (i) $GL(n, \mathbb{C}) \subset \mathbb{C}^{n^2}$

$GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ open subsets $\Rightarrow$ They are Lie group since

$g \rightarrow g^{-1}$ is analytic (holomorphic in the 1st case then G is called a complex Lie group).

(ii) $SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$

$$SL(n,\mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det(A) = 1\} \quad \text{Special linear group}$$

$f(A) := \det(A) - 1$

Non-degenerate smooth function $\Rightarrow$ $SL(n, \mathbb{R})$ is a Lie group.

$SL(n, \mathbb{R})$ is a closed subgroup of $GL(n, \mathbb{R})$.

($\exists$ a general thm: Any closed subgroup of a Lie group is a Lie subgroup)

Def^n 1.b $\mathfrak{g}$ (over $\mathbb{R}$, $\mathbb{C}$) a vector space of dim n is called a Lie algebra if $\exists$ a bracket $[x, y]$

Satisfies (i) it is bilinear $[ax+by, z] = a[x\ z] + b[y\ z]$

(ii) $[v, w] = -[w, v]$

(iii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

E.g. (i) $(\mathbb{R}^3, \times)$ $\quad [x, y] := x \times y$

$\times$ = Cross product

$$x \times (y \times z) + z \times (x \times y) + y \times (z \times x) = 0 \qquad \big((x \times y) \times z + (z \times x) \times y + (y \times z) \times x = 0 \big)$$

$$\left.\begin{aligned} (x \times y) \times z &= +(\langle z\ x \rangle y - \langle z, y \rangle x) \\ (z \times x) \times y &= +(\langle y\ z \rangle x - \langle y\ x \rangle z) \\ (y \times z) \times x &= +(\langle x\ y \rangle z - \langle x\ z \rangle y) \end{aligned}\right\} \Rightarrow \text{ Jacobi identity.}$$

(ii) $\mathfrak{g} = \mathcal{X}(M)$ — smooth vector field of M.

$$X = X^i \frac{\partial}{\partial x^i} \qquad [X, Y]f = XYf - YXf$$

$$X^i \frac{\partial}{\partial x^i}\left(Y^h \frac{\partial f}{\partial x^h} \right) - Y^i \frac{\partial}{\partial x^i}\left(X^h \frac{\partial f}{\partial x^h} \right)$$

$$= \left[\left(X^i \frac{\partial Y^h}{\partial x^i} \right) \frac{\partial}{\partial x^h} - \left(Y^i \frac{\partial X^h}{\partial x^i} \right) \frac{\partial}{\partial x^h} \right] f$$

$$\Rightarrow [X, Y]^h = \sum X^i \frac{\partial Y^h}{\partial x^i} - \sum Y^i \frac{\partial X^h}{\partial x^i} \quad \text{in local coordinate.}$$

2 Lie Group $\xrightarrow{\quad}$ Lie algebra

taking derivative

(i) $\mathfrak{g} \simeq G_e \left(\simeq T_e G\right)$ ↑ the tangent space at identity.

is called $\mathrm{Lie}(G)$ — the Lie algebra of G.

$\forall\, x, y \in G_e$

Let X, Y be the left invariant v.f. generated by x. y

$X(g) \doteq (L_g)_*(x)$ $\qquad$ $Y(g) := (L_g)_*(x)$

$L_g : G \to G$ $\qquad$ $h \longrightarrow gh$.

If $(L_g)_*(X(\sigma)) = X(g\sigma)$ $\quad$ X is called left invariant

Clearly the $X(g)$, $Y(g)$ defined above is left invariant since

$(L_g)_* X(\sigma) = (L_g)_*((L_\sigma)_*)(x) = (L_{g\sigma})_*(x) = X(g\sigma)$

Then $[x. y] := [X. Y]\big|_e$

Since $[[X\ Y]\ Z] + [[Y, Z].\ X] + [[Z.\ X].\ Y] = 0$

$\Rightarrow$ $\mathfrak{g}$ is a Lie algebra.

(ii) The $[\ \ ]$ can be obtained by taking 2nd derivative of the product structure.

To explain this we need the exponential map.

Let X be a left invariant v.f. it generates a 1-parameter family of diffeomorphism $\varphi(t,\cdot)$ by solving the ODE

$$\begin{cases} \dfrac{d\varphi}{dt} = X(\varphi) \\ \varphi(0) = a \end{cases} \qquad \left(\begin{cases} \dfrac{dx^1}{dt} = X^1(x) & x^1(0) = x_0^1 \\ \vdots & \vdots \\ \dfrac{dx^n}{dt} = X^n(x) & x^n(0) = x_0^n \end{cases} \quad (x_0^1 \cdots x_0^n) = P_0 \in G \right)$$

gives $\varphi(t, a)$ $\qquad \varphi(0, a) = a$

$a \to \varphi(t, a)$ is a diffeomorphism.

(From now on we identify $x \in \mathfrak{g} \simeq G_e$ with X - left invariant v.f).

$\exp(tX) := \varphi(t, e)$

Since

$$\begin{cases} \dfrac{d}{dt}\left(\varphi(t+s, e)\right) = X(\varphi(t+s, e)) \\ \varphi(0+s, e) = \varphi(s, e) \end{cases}$$

&

$$\begin{cases} \dfrac{d}{dt}\left(\varphi(s,e)\varphi(t,e)\right) = \left[L_{\varphi(s,e)}\right]_* X(\varphi(t,e)) \\ \qquad\qquad = X(\varphi(s,e)\cdot\varphi(t,e)) \\ \varphi(s,e)\underbrace{\varphi(0,e)}_{=e} = \varphi(s,e) \end{cases}$$

$\Rightarrow$ By the uniqueness of ODE solution

$$\varphi(t+s, e) = \varphi(s, e)\varphi(t, e)$$

Namely $\exp((t+s)X) = \exp(sX)\cdot\exp(tX)$.

This defines $\exp(tX)$. $\quad \exp(X) := \exp(tX)\big|_{t=1}$

~~For~~

Similarly $\quad \varphi(t, a) = a \cdot \varphi(t. e) \quad \forall a \in G$

$$= \cancel{R_a(\varphi(t.e))}\ R_{\varphi(t\,e)}(a)$$

Right multiplication

~~Re~~ Recall the result:

$$[X, Y](x) = \frac{d}{dt}\Big|_{t=0} \left[d\varphi_{-t} \left(Y_{\varphi_t(x)} \right) \right]$$

(Thm 4 of Ch 2.12 of Matsushima] $\quad \varphi_t$ being the 1-parameter family of diffeomorphism generated by X.

Apply to our case X, Y

$$[X, Y](e) = \frac{d}{dt}\Big|_{t=0} \left[d\varphi_{-t} \left(Y_{\varphi_t(e)} \right) \right]$$

$\varphi_{-t}(\cancel{\varphi_t(e)}\ a)$

$'' \varphi(-t, a)$

$= a\ \varphi(-t, e)$

$$= \frac{d}{dt}\Big|_{t=0} \left(dR_{\exp(-tX)} \cdot \left(Y_{\varphi_t(e)} \right) \right)$$

$$Y_{\varphi_t(e)} = dL_{\varphi_t(e)}(Y_e)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(Ad\left(\exp(tX)\right)(Y_e) \right) \qquad (*)$$

Recall that Ad is the adjoint representation which can be given by da_h

$a_h : G \to G$ is $\quad g \to hgh^{-1}$

$da_h : \mathfrak{g} \to \mathfrak{g}$ $\qquad X \in \mathfrak{g}$ $\qquad \exp(tX)$ is the curve passing e with tangent X.

$$da_h(X) = \frac{d}{dt}\Big|_{t=0} a_h(\exp(tX))$$

$$= \frac{d}{dt}\Big|_{t=0} h \exp(tX) h^{-1}$$

Hence

$$\mathrm{Ad}_h(Y) = \frac{d}{ds}\Big|_{s=0} h \exp(sY) h^{-1}$$

$$\Rightarrow \quad \mathrm{Ad}_{\exp(tX)}(Y) = \frac{d}{dt}\Big|_{t=0}\left(\frac{d}{ds}\Big|_{s=0} \exp(tX)\exp(sY)\exp(-tX)\right)$$

$$= \frac{\partial^2}{\partial t \partial s}\Big|_{(0,0)} \left(\exp(tX)\exp(sY)\exp(-tX)\right) \qquad (1)$$

$$\mathrm{Ad} : G \to GL(n, \mathfrak{g})$$
$$h \to \mathrm{Ad}_h$$

$d\mathrm{Ad}$ is denoted as $\mathrm{ad} : \mathfrak{g} \to \underline{\mathfrak{gl}(n, \mathfrak{g})}$

Lie algebra of general linear transformations

$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$.

Claim $\mathrm{ad}_X(Y) = [X\ Y]$

This follows from $(*)$ & the above discussion!

③ Lie-algebra $\xrightarrow[\text{2 integrations}]{}$ Lie group – I

To describe this process we need some preperations.

$$\exp : G_e \longrightarrow G$$
$$X \longrightarrow \exp(X)$$

For the curve $c(t) := tX \subset G_e \simeq \mathfrak{g}$ $\qquad c'(0) = X$

$\tilde{c}(t) = \exp(tX)$ is the image curve $\qquad \tilde{c}'(0) = X$

$\Rightarrow (d\exp)_e = id$

Hence $\exp : G_e \longrightarrow G$ is a local diffeomorphism by the inverse function theorem

$\Rightarrow$ we may use it to give a local coordinate

$\exp(tX)$ is given by $\underbrace{t.(x^1, \dots x^n)}_{\substack{\shortparallel \\ tX}}$ if $X = \sum x^i \frac{\partial}{\partial x_i}$

– by abuse notation

The product $\psi : G \cancel{\times} G \to G$ near (e, e)

Can be described with a vector-valued function ψ depends on x & y

$\in \mathbb{R}^n \simeq \mathfrak{g}$ $\quad \in \mathbb{R}^n \simeq \mathfrak{g}$

$$\begin{cases} \psi(x, 0) = \psi(0, x) = x \\ \psi(x, \varphi(x)) = 0 \\ \\ \psi(x, \psi(y, z)) = \psi(\psi(x, y), z) \end{cases}$$

$\xi, \eta \in G_e$ we may define the Lie bracket also using the Taylor expansion coefficient in the local coordinate

$$\psi^\alpha(x, y) = x^\alpha + y^\alpha + b^\alpha_{\beta\gamma} x^\beta y^\gamma + h^\alpha_{ijk} x^i y^j y^k + g^\alpha_{ijk} x^i x^j y^k$$

+ higher order

No $x^\beta x^\gamma$ term since $\psi^\alpha(x, o) = x^\alpha$

We define the Lie bracket with

$$[\xi, \eta]^\alpha \doteq (b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta}) \xi^\beta \eta^\gamma \qquad (*)$$

Namely $C^\alpha_{\beta\gamma} := b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta}$.

$$[[\xi, \eta], z] = C^s_{\beta\gamma} \xi^\beta \eta^\gamma z^t C^\alpha_{st} = C^\alpha_{st} C^s_{\beta\gamma} \xi^\beta \eta^\gamma z^t$$

$(\beta\, \gamma.\, t) \to (\cancel{\gamma}\, t\, \beta)$

$\to (t, \beta, \gamma)$

Jacobi identity $\Leftrightarrow$

$$C^\alpha_{st} C^s_{\beta\gamma} + C^\alpha_{s\beta} C^s_{\gamma t} + C^\alpha_{s\gamma} C^s_{t\beta} = 0 \qquad (**)$$

Now we need to check $(*)$ so-defined also satisfies the Jacobi identity.

This can be done by direct calculation using the expansion of

$$\psi^\alpha(x, y) \quad \& \quad \psi(\psi(x, y), z) = \psi(x, \psi(y, z)).$$

However it can also be done by the following consideration via equation (1) in the 1st-kind coordinate.

In the coordinate of the 1st kind $\quad \underset{x(t)}{\exp(tX)} = tX$

$\exp(-tX) = -tX$

$\exp(\tau Y) = \tau Y.$

Hence $\quad \underbrace{x(\tau)}\, y(t)\, x(\tau)^{-1}$ is given by

$$\left(\tau X^\alpha + tY^\alpha + b^\alpha_{\beta\gamma}\, \tau \cdot t\, X^\beta Y^\gamma\right) - \tau X^\alpha + b^\alpha_{\beta\gamma}\left(\tau X^\beta + tY^\beta + b^\beta_{\mu\nu}\tau t X^\mu Y^\nu\right)(-\tau X)^\gamma$$

+ high in τ, t

$$= tY^\alpha + \left(b^\alpha_{\beta\gamma} X^\beta Y^\gamma\right)(\tau t) - \left(b^\alpha_{\beta\gamma} Y^\beta X^\gamma\right)(\tau t)$$

By (1) $\Rightarrow \quad [X, Y]^\alpha = \underbrace{(b^\alpha_{\beta\gamma} - b^\alpha_{\gamma\beta})}_{= C^\alpha_{\beta\gamma}} X^\beta Y^\gamma$

Hence $C^\alpha_{\beta\gamma}$ does satisfy the Jacobi identity.

(4) Locally one may solve a PDE system if the compatibility conditions hold.

Locally:
$$\begin{cases} \dfrac{\partial u^i}{\partial x_j} = \varphi^i_j(u_1, \dots u_n, x_1 \dots x_n) & 1 \le i, j \le n \\ u^i(x_0) = u^i_0, \quad (u(x_0) = u_0) \end{cases}$$

Can be solved if the compatibility conditions

$$\frac{\partial \varphi^i_k}{\partial u^l}\varphi^l_j + \frac{\partial \varphi^i_k}{\partial x_j} = \frac{\partial \varphi^i_j}{\partial u^l}\varphi^l_k + \frac{\partial \varphi^i_j}{\partial x^k} \quad \text{holds}$$

$\forall\, i, j, k$

Repeated indices means summation

(1849-1917)

This is a Kind of Frobenius theorem!

We solve the PDE by ODE. This perhaps is the original approach!

$C(t)$ — which is given by $\frac{dx^i}{dt}$ as its tangent vector.

$$\begin{cases} \dfrac{du^i}{dt} = \varphi^i_j(u, x)\dfrac{dx^j}{dt} \\ u^i(0) = u^i_0 \end{cases}$$

We only need to show that it is independent of the choice of path.

$C(\alpha, t)$ is a family of path from x_0 to x (by simply-connected-ness of the local region)

$C(0, t) = C(t)$

$C(t, 1) = \tilde{C}(t)$ another curve

Consider the equation: $\dfrac{\partial u^i}{\partial \alpha} = \varphi^i_j(u, x)\dfrac{\partial x^j}{\partial \alpha}$ $\qquad \left.\dfrac{\partial x^j}{\partial \alpha}\right|_{t=1} = 0$

Let $\quad \varepsilon^i = \dfrac{\partial u^i}{\partial \alpha} - \varphi^i_j(u, x)\dfrac{\partial x^j}{\partial \alpha}$

$$\frac{\partial \varepsilon^i}{\partial t} = \frac{\partial^2 u^i}{\partial t \partial \alpha} - \frac{\partial}{\partial t}\left(\varphi^i_j(u, x)\right)\frac{\partial x^j}{\partial \alpha} - \varphi^i_j(u, x)\frac{\partial^2 x^j}{\partial \alpha \partial t} \qquad (4.1)$$

$$\frac{\partial^2 u^i}{\partial t \partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{\partial u^i}{\partial t}\right) = \frac{\partial}{\partial \alpha}\left(\varphi^i_j(u, x)\frac{\partial x^j}{\partial t}\right)$$

$$= \frac{\partial \varphi^i_j}{\partial u^l}\frac{\partial u^l}{\partial \alpha}\frac{\partial x^j}{\partial t} + \frac{\partial \varphi^i_j}{\partial x^l}\frac{\partial x^l}{\partial \alpha}\frac{\partial x^j}{\partial t} + \varphi^i_j(u, x)\frac{\partial^2 x^j}{\partial \alpha \partial t}$$

& $\dfrac{\partial \varphi^i_j(u,x)}{\partial t} = \dfrac{\partial \varphi^i_j}{\partial u^l}\dfrac{\partial u^l}{\partial t} + \dfrac{\partial \varphi^i_j}{\partial x^l}\dfrac{\partial x^l}{\partial t}$

plugging these into (4.1) $\Rightarrow$

$$\frac{\partial \varepsilon^i}{\partial t} = \frac{\partial \varphi^i_j}{\partial u^l}\underbrace{\frac{\partial u^l}{\partial \alpha}}_{\varepsilon^l + \varphi^l_k \frac{\partial x^k}{\partial \alpha}}\frac{\partial x^j}{\partial t} + \frac{\partial \varphi^i_j}{\partial x^l}\frac{\partial x^l}{\partial \alpha}\frac{\partial x^j}{\partial t} - \frac{\partial \varphi^i_j}{\partial u^l}\frac{\partial u^l}{\partial t}\frac{\partial x^j}{\partial \alpha} - \frac{\partial \varphi^i_j}{\partial x^l}\frac{\partial x^l}{\partial t}\frac{\partial x^j}{\partial \alpha}$$

$$= \frac{\partial \varphi^i_j}{\partial u^l}\left(\varepsilon^l + \varphi^l_k \frac{\partial x^k}{\partial \alpha}\right)\frac{\partial x^j}{\partial t} + \frac{\partial \varphi^i_j}{\partial x^k}\frac{\partial x^k}{\partial \alpha}\frac{\partial x^j}{\partial t}$$

$$- \frac{\partial \varphi^i_j}{\partial u^l}\cancel{\frac{\partial u^l}{\partial t}}\left(\varphi^l_k \frac{\partial x^k}{\partial t}\right)\frac{\partial x^j}{\partial \alpha} - \frac{\partial \varphi^i_j}{\partial x^k}\frac{\partial x^k}{\partial t}\frac{\partial x^j}{\partial \alpha}$$

$$= \frac{\partial \varphi^i_j}{\partial u^l}\varepsilon^l + \left(\underbrace{\left[\frac{\partial \varphi^i_j}{\partial u^l}\varphi^l_k + \frac{\partial \varphi^i_j}{\partial x^k}\right] - \left[\frac{\partial \varphi^i_k}{\partial u^l}\varphi^k_j + \frac{\partial \varphi^i_k}{\partial x^j}\right]}_{=0}\right)\frac{\partial x^j}{\partial t}\frac{\partial x^k}{\partial x^\alpha}$$

due to the compatibility condition

$$= \frac{\partial \varphi^i_j}{\partial u^l}\varepsilon^l$$

$\varepsilon^i(0,\alpha) = \varepsilon^i(1,\alpha) = 0 \Rightarrow \varepsilon^l \equiv 0$

Hence we have $\dfrac{\partial u^i}{\partial \alpha} = \varphi^i_j(u,x)\dfrac{\partial x^j}{\partial \alpha}$

$\Rightarrow$ At $t=1$ $\quad u^i(1,0) = u^i(1,1)$

This proves that the ODE solution is independent of the choice of the path $\Rightarrow$ It solves the PDE.

(5) Derive the PDE for the product function $\psi(x, y)$.

Define $V^{\alpha}_{\beta}(x) \doteq \left.\dfrac{\partial \psi^{\alpha}(x,y)}{\partial x^{\beta}}\right|_{y=\varphi(x)}$.

$$\psi^{\alpha}(x, y) = x^{\alpha} + y^{\alpha} + b^{\alpha}_{\beta\gamma} x^{\beta} y^{\gamma} + \cdots$$

$$\frac{\partial \psi^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + b^{\alpha}_{\beta\gamma} y^{\gamma}$$

Now let $y = x^{-1}$ namely $y = -x$ in the coordinate of 1st kind

$$\Rightarrow \quad \left.\frac{\partial \psi^{\alpha}}{\partial x^{\beta}}\right|_{y=\varphi(x)} = \delta^{\alpha}_{\beta} - b^{\alpha}_{\beta\gamma} x^{\gamma}$$

Hence it is invertible if x is close to e.

Lemma: $\begin{cases} V^{\alpha}_{\beta}(\psi) \dfrac{\partial \psi^{\beta}_{\alpha}(x, y)}{\partial x^{\gamma}} = V^{\alpha}_{\gamma}(x) \\ \psi(0, y) = y \end{cases}$

Pf:

$$LHS = \left.\frac{\partial \psi^{\alpha}}{\partial x^{\beta}}\right|_{\substack{x=\psi(x,y)\\ y=\varphi(\psi(x,y))}} \cdot \frac{\partial \psi^{\beta}}{\partial x^{\gamma}}(x, y)$$

$$= \frac{\partial \psi^{\alpha}}{\partial x^{\gamma}}(\psi(x, y)\ \ z) \qquad z = \varphi(\psi(x, y))$$

$$= \left.\frac{\partial \psi^{\alpha}}{\partial x^{\gamma}}\left(x, \underline{\psi(y, z)}\right)\right|_{z=\varphi(\psi(x,y))} \qquad \text{use } \psi(\psi(x\ y)\ z) = \psi(x, \psi(y, z))$$

$$= \frac{\partial \psi^{\alpha}}{\partial x^{\gamma}}(x, w) = V^{\alpha}_{\gamma}(x)$$

$$w = \psi(y\ z) = y \cdot (x \cdot y)^{-1} = x$$

The goal is solve
$$\begin{cases} \dfrac{\partial \psi^\beta}{\partial x^\gamma}(x, y) = u^\beta_\alpha(\psi)\, v^\alpha_\gamma(x) & \text{(PDE)} \\ \psi(0, y) = y \end{cases}$$

(u^β_α) is the inverse of (v^β_α)

is the PDE we want to solve. y is the parameter.

The compatibility condition of this PDE is

$$\left(\frac{\partial v^j_i}{\partial x^k} - \frac{\partial v^j_k}{\partial x^j}\right) = \sum C^i_{\gamma\delta}\, v^\gamma_j v^\delta_k \qquad (4.2)$$

The question is how to find v^j_i satisfy (4.2).

This can be done by solving

$$\begin{cases} \dfrac{dw^i_j}{dt} = \delta^i_j + C^i_{\alpha\beta}\, a^\alpha w^\beta_j & a \in \mathbb{R}^n \qquad (4.3) \\ w^i_j(0) = 0 \end{cases}$$

& Let $v^i_j(x) = w^i_j(1, x)$

To check (4.2) holds one needs the Jacobi-identity.

The construction is backwards. 1st Solve (4.3) $\Longrightarrow v^i_j(x)$

check (4.2) holds for $v^i_j(x)$. Then solve the (PDE)

to get $\psi(x, y)$. Finally check ψ satisfies the associativity.