shall henceforth call this decomposition the **root space decomposition** (relative to $\mathfrak{h}$, to be more precise). It used to be called the Cartan decomposition but we are using that name for something else in §3.3.

Before we proceed to the proof of Theorem 3.5 we give a concrete description for each classical Lie algebra (see Example 2.2). The root space decompositions in these cases are obtained by elementary computation. The situations are not so simple for simple Lie algebras of exceptional types.

EXAMPLE 3.4. (i) $\mathfrak{g}_C = \mathfrak{sl}(l+1, C)$, $l \geq 1$, $G = SU(l+1)$,

$$\mathfrak{h} = \left\{ \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_{l+1}) \mid \lambda_i \in C, \sum_{i=1}^{l+1} \lambda_i = 0 \right\}.$$

Thus $\dim_C \mathfrak{h} = l$. This $\mathfrak{h}$ is the Cartan subalgebra obtained from the standard maximal torus subgroups mentioned in Example 2.2. For each $\{i, j\}$ such that $1 \leq i, j \leq l+1$, let E_{ij} be the matrix whose (i, j)-component is 1 while all others are 0. By writing

$$H = \sum_{i=1}^{l+1} \lambda_i(H) E_{ii}$$

we consider each λ_i as an element of $\mathfrak{h}^*$. Then for any $H \in \mathfrak{h}$ we get

$$[H, E_{ij}] = (\lambda_i - \lambda_j)(H) E_{ij}.$$

Therefore by setting $\alpha_i = \lambda_i - \lambda_{i+1}$, where $1 \leq i \leq l$, we find that $\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$ $(i < j)$ are roots and

$$\Delta = \{\lambda_i - \lambda_j \mid i \neq j\}$$

as we see from the dimension relations. Also in this case, we may take $E_{\lambda_i - \lambda_j} = E_{ij}$ $(i \neq j)$. Thus the root space decomposition is nothing but the decomposition of $\mathfrak{sl}(l+1, C)$ as the direct sum of the space of all diagonal matrices $\mathfrak{h}$, the vector space of all upper triangular matrices and that of all lower triangular matrices.

(ii) $\mathfrak{g}_C = \mathfrak{o}(n, C)$, $(n \geq 3)$, $G = SO(n, R)$. We consider the cases where n = even $(= 2l)$ and n = odd $(= 2l+1)$ separately.

First, let $n = 2l$. In order to find a simpler expression for the Cartan subalgebra, we transform $\mathfrak{g}_C$ by the following matrix

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} I_l & \sqrt{-1} I_l \\ I_l & -\sqrt{-1} I_l \end{bmatrix} \in U(2l).$$

Namely, we get $\mathfrak{g}'_C = L \mathfrak{g}_C L^{-1}$, isomorphic to $\mathfrak{g}_C$, such that

$$\mathfrak{g}'_C = \{X' \in M(2l, C) \mid {}^t X' S + S X' = 0\},$$

where

$$S = \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix}.$$

In more concrete form we have

$$\mathfrak{g}'_C = \left\{ \begin{bmatrix} X & Y \\ Z & -{}^t X \end{bmatrix} \middle| X, Y, Z \in M(l, C), {}^t Y = -Y, {}^t Z = -Z \right\}.$$

Now set

$$\mathfrak{h}' = \{\operatorname{diag}(\lambda_1, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l) \mid \lambda_i \in C \ (1 \leq i \leq l)\}.$$

This means that the maximal torus subgroup T^l of $SO(n, R)$ in Example 2.2 can be diagonalized by L into

$$T' = \{\operatorname{diag}(t_1, \dots, t_l, t_1^{-1}, \dots, t_l^{-1}) \mid t_i \in C, |t_i| = 1 \ (1 \leq i \leq l)\}.$$

Accordingly, we have

$$\mathfrak{t}' = \{\operatorname{diag}(\lambda_1, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l) \mid \lambda_i \in C, \operatorname{Re}(\lambda_i) = 0 \ (1 \leq i \leq l)\}.$$

Now the decomposition of $\mathfrak{g}'_C$ relative to $\mathfrak{h}'$ can be given as follows. For this purpose, if we set

$$E_{\lambda_i + \lambda_j} = \begin{bmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{bmatrix},$$

$$E_{-\lambda_i - \lambda_j} = \begin{bmatrix} 0 & 0 \\ E_{ij} - E_{ji} & 0 \end{bmatrix} \quad (1 \leq i < j \leq l),$$

$$E_{\lambda_i - \lambda_j} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \quad (i \neq j),$$

we can easily obtain the following relations. Namely, for an arbitrary element of $\mathfrak{h}'$ in the form

$$H = H(\lambda_1, \dots, \lambda_l) = \operatorname{diag}(\lambda_1, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l),$$

we get

$$[H, E_{\pm\lambda_i \pm \lambda_j}] = (\pm\lambda_i \pm \lambda_j) E_{\pm\lambda_i \pm \lambda_j}.$$

We see therefore that Δ consists of $\pm\lambda_i \pm \lambda_j$ by considering the dimensions of $\mathfrak{g}'_C$ and $\mathfrak{h}'$. These roots are linear combinations of the l roots $\{\alpha_1, \dots, \alpha_l\}$; $\alpha_i = \lambda_i - \lambda_{i+1}$ $(1 \leq i \leq l-1)$, and $\alpha_l = \lambda_{l-1} + \lambda_l$.

We now discuss the case $n = 2l+1$. In a similar fashion we take

$$L' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_l & \sqrt{-1} I_1 \\ 0 & I_l & -\sqrt{-1} I_1 \end{bmatrix} \in U(2l+1),$$

and get a Lie algebra isomorphic to $\mathfrak{g}_C$

$$\mathfrak{g}'_C = \{X' \in M(2l+1, C) \mid {}^tX'S' + S'X' = 0\}$$
$$= \left\{ X' = \begin{bmatrix} 0 & a & b \\ -{}^tb & X & Y \\ {}^ta & Z & -{}^tX \end{bmatrix} \middle| X, Y, Z \text{ as before, } a, b \in C^l \right\},$$

where

$$S' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix}.$$

We obtain its Cartan subalgebra $\mathfrak{h}'$ in the form

$$\mathfrak{h}' = \{\mathrm{diag}(0, \lambda_1, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l) \mid \lambda_i \in C \quad (1 \le i \le l)\}$$

as well as

$$T' = \{\mathrm{diag}(1, t_1, \dots, t_l, -t_1^{-1}, \dots, -t_l^{-1}) \mid t_i \in C, |t_i| = 1 \quad (1 \le i \le l)\},$$

$$\mathfrak{t}' = \{\mathrm{diag}(0, \lambda_1, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l) \mid \lambda_i \in C, \mathrm{Re}(\lambda_i) = 0 \quad (1 \le i \le l)\}.$$

As for the root space decomposition of $\mathfrak{g}'_C$ relative to $\mathfrak{h}'$ we have a situation different from the case where n is even and need the following notation. For each $1 \le i \le l$, denote by E_{λ_i} the element $X' \in \mathfrak{g}'_C$ in the form above where $a = (0, \dots, 1, \dots, 0)$ with 1 as the ith component, $b = 0$, $X = Y = Z = 0$, and by $E_{-\lambda_i}$ the element of the same form where a and b are interchanged. Then for

$$H = H(\lambda_1, \dots, \lambda_l) = \mathrm{diag}(0, \lambda, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l),$$

we get

$$[H, E_{\pm\lambda_i}] = \pm\lambda_i E_{\pm\lambda_i}.$$

Again from the dimension considerations, we see that the root system Δ consists of $\pm\lambda_i \pm \lambda_j$, $\pm\lambda_i$. Furthermore if we set

$$\alpha_i = \lambda_i - \lambda_{i+1} \quad (1 \le i \le l-1), \qquad \alpha_l = \lambda_l,$$

then all roots can be expressed as linear combinations of the l roots $\{\alpha_1, \dots, \alpha_l\}$. (Note that if $n = 2$ in the discussions above then $\mathfrak{o}(2, C)$ is 1-dimensional and commutative.)

(iii) $\mathfrak{g}_C = \mathfrak{sp}(l, C)$, $(l \ge 1)$, $G = Sp(l)$. We take as Cartan subalgebra

$$\mathfrak{h} = \{\mathrm{diag}(\lambda_1, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l) \mid \lambda_i \in C\, (1 \le i \le l)\}.$$

In order to get the root space relative to this $\mathfrak{h}$, we introduce the following matrices:

$$E_{\lambda_i+\lambda_j} = \begin{bmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{bmatrix}, \qquad E_{-\lambda_i-\lambda_j} = \begin{bmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{bmatrix},$$
$$E_{\lambda_i-\lambda_j} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix} \qquad (i \ne j).$$

Then for

$$H = H(\lambda_1, \dots, \lambda_l) = \mathrm{diag}(\lambda, \dots, \lambda_l, -\lambda_1, \dots, -\lambda_l),$$

we get

$$[H, E_{\pm\lambda_i\pm\lambda_j}] = (\pm\lambda_i \pm \lambda_j) E_{\pm\lambda_i\pm\lambda_j}.$$

Thus for the same reason as before we have $\Delta = \{\pm\lambda_i \pm \lambda_j\}$. By setting $\alpha_i = \lambda_i - \lambda_{i+1}$ $(1 \le i \le l-1)$ and $\alpha_l = 2\lambda_l$, we find that all roots are linear combinations of $\{\alpha_1, \cdots, \alpha_l\}$.

REMARK 3.4. The Cartan subalgebras written down in Example 3.4 are nothing but the complexifications $\mathfrak{t}_C$ of the real commutative Lie subalgebras $\mathfrak{t}$ for the maximal torus subgroups T enumerated in Example 2.2. The only exception is the case (ii) for which the conjugate $\mathfrak{t}'_C$ is used. Their advantage is that the forms are simple and easy to treat; they are often called the **standard Cartan subalgebras**. For each type (i), (ii), (iii) we can find special roots $\{\alpha_1, \cdots, \alpha_l\}$, where l is the rank, such that all other roots are their linear combinations with integral coefficients. These special roots are called the fundamental roots. This phenomenon actually occurs to the root space decomposition of complex semisimple Lie algebras in general. We shall come back to this matter in Theorem 3.6.

Now we shall prove Theorem 3.5. Since the subset $\mathrm{ad}(\mathfrak{h})$ of $\mathfrak{gl}(\mathfrak{g}_C)$ is a commutative set consisting of semisimple linear transformations, we can simultaneously diagonalize it and thus find a decomposition $\mathfrak{g}_C = \mathfrak{g}_0 \oplus \sum_{\alpha\ne 0} \mathfrak{g}_\alpha$. Since $\mathfrak{h}$ is a maximal commutative Lie subalgebra, we have (i). In order to prove (ii), we study the structure of the root space decomposition. First of all, we remark that, for any $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ we get by Jacobi's identity

$$[H, [X, Y]] = 0 \quad \text{for all } H \in \mathfrak{h}.$$

It follows that $[X, Y] \in \mathfrak{g}_0 = \mathfrak{h}$. Thus we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$. This proves the second statement of (ii).

Now let $\alpha, \beta \in \Delta$ and $\alpha + \beta \ne 0$. Suppose $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$. Then we get for $H \in \mathfrak{h}$

$$[H, [X, Y]] = (\alpha + \beta)(H)[X, Y].$$