(1)

Examples:

(a) $u(n) := \{ a \mid a + a^* = 0, \quad a^* := \overline{a^{tr}} \}$

$su(n) := \{ a \in u(n), \ tr(a) = 0 \}$

It is easy to see the Complexification of $sl(n, \mathbb{R})$ is $sl(n, \mathbb{C})$

Span$\{ \lambda a,$ with $\lambda \in \mathbb{C}, \ a \in sl(n, \mathbb{R}) \}$ clearly $= sl(n, \mathbb{C})$

the Standard basis of $sl(n, \mathbb{R})$ is e_{ij} $i \neq j$

& $e_{ii} - e_{i+1\, i+1}$ $1 \leq i \leq n-1$

They Spans $sl(n, \mathbb{C})$ over $\mathbb{C}$.

But the Complexification of $su(n)$ is also $sl(n, \mathbb{C})$.

1st by the dimension counting $\dim_{\mathbb{R}}(su(n)) = n^2 - 1$

Complexification makes it $\dim_{\mathbb{C}} = n^2 - 1$ $\left(= \dim_{\mathbb{C}}(sl(n, \mathbb{C})) \right)$

Moreover $\forall$ $b \in sl(n, \mathbb{C})$

$$b = \frac{b + b^*}{2} + \frac{b - b^*}{2} = \underset{\in su(n)}{a} + \underset{\in su(n)}{-i\,\tilde{a}}$$

Let $a = \frac{b - b^*}{2}$, $a \in su(n)$

$$\left.\begin{array}{l} \tilde{a} = \dfrac{i(b + b^*)}{2} \\ \tilde{a} + \tilde{a}^* = \dfrac{ib + ib^* - i(b^* + b)}{2} = 0 \qquad tr(\tilde{a}) = 0 \end{array}\right\} \Rightarrow \tilde{a} \in su(n)$$

$\Rightarrow sl(n, \mathbb{C}) \xhookrightarrow{i} su(n) \otimes \mathbb{C}$. $\Longrightarrow sl(n, \mathbb{C}) \simeq su(n) \otimes \mathbb{C}$.

Namely both $sl(n, \mathbb{R})$, — real Lie algebra, $SL(n, \mathbb{R})$ ← Noncompact Lie group

& $su(n)$ — $SU(n)$ ← real compact Lie group $\{ SU(2) \simeq S^3$

Complexify into $sl(n, \mathbb{C})$.

[I made mistake when I mentioned 'hiking', which was on Dixon lake about a year ago]

$u(n)$ - complexifies $gl(n, \mathbb{C})$, as $gl(n, \mathbb{R})$.

$\dim_{\mathbb{R}}(\ ) = n^2$ One needs this to understant curvature on Complex manifolds

(b) $o(p, q)$ $p+q=n$ preserves the form $B(x, x) = -\sum_{i=1}^{p} |x_i|^2 + \sum_{j=p+1}^{n} |x_j|^2$

$Q_p = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix}$ is the matrix form $B(x, y) = x^t Q_p y$

$X \in o(p, q)$ if $B(Xx, y) + B(x, Xy) = 0 \quad \forall x, y \in \mathbb{R}^n$

$$\Rightarrow \quad x^t (X^t Q_p + Q_p X) y = 0$$

Namely if $X = \begin{pmatrix} A, & B \\ C & D \end{pmatrix} \in o(p, q) \Rightarrow A \in o(p)$, $D \in o(q)$

$B^t = C$

Namely it has the form $\begin{pmatrix} A, & B \\ B^t & D \end{pmatrix}$ $A \in o(p)$, $D \in o(q)$.

Then $o(p, q) \otimes \mathbb{C} = \left\{ \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, \ A + A^t = 0, \ D + D^t = 0 \quad A, B, D \text{ complex} \right\}$

$\phi_J : X \to J_p X J_p^{-1}$ $J_p = \begin{pmatrix} \sqrt{-1} I_p & 0 \\ 0 & I_q \end{pmatrix}$

$\Rightarrow \phi_J$ clearly is a Lie algebra isomorphism

$$\phi([X, Y]) = [\phi(X), \phi(Y)]$$

On the other hand, $\phi \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} = \begin{pmatrix} \sqrt{-1} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \begin{pmatrix} -\sqrt{-1} I, & 0 \\ 0 & I \end{pmatrix}$

$$= \begin{pmatrix} \sqrt{-1}A & \sqrt{-1}B \\ B^t & D \end{pmatrix} \begin{pmatrix} -\sqrt{-1}I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & \sqrt{-1}B \\ -\sqrt{-1}B^t & D \end{pmatrix}$$

Namely $\phi(X) + \phi(X)^t = 0 \iff \phi(X) \in o(n, \mathbb{C})$.

Summarizing $O(p,q)$ $0 \le p < [\frac{n}{2}]+1$ all complexify into $o(n, \mathbb{C})$.

Only one of them, namely $o(n)$ is compact.

② A general result

Ziller: 1.39: If g is complex Lie algebra, $(g_{\mathbb{R}})_{\mathbb{C}} = g \oplus \bar{g}$

Not as Lie algebra ideals but vector spaces

$J((u,v)) = (-v, u)$ the meaning is $\underset{\substack{\| \\ u+jv}}{(u,v)} \cong g \oplus g$

as vectorspace.

But $j(u+jv) = -v+ju$ Hence multiply j means $u+jv \to -v+ju$

The Lie algebra structure on $g \oplus g$ is defined to be

$$[(u,v),(x,y)] = ([u,x]-[v,y],\ [u,y]+[v,x]) \qquad (*)$$

That is the Lie algebra on $g \oplus g$, when we complexify $g_{\mathbb{R}}$

The original multiplication is I $\quad I((u,v)) = (Iu, Iv)$.

$$(J \cdot I)(u,v) = (-Iv, Iu) = I \cdot J(u,v)$$

$L = J \cdot I \qquad L^2 = id$.

$$f_-(u) \doteq \tfrac{1}{2}(u, -Iu)$$

$$f_+(u) \doteq \tfrac{1}{2}(u, Iu).$$

$$(u, v) = \left(\frac{u - Iv}{2}, \underbrace{\frac{Iu - I(Iv)}{2}}\right) + \left(\underbrace{\frac{u+Iv}{2}}_{w'}, \underbrace{-I\left(\frac{u+Iv}{2}\right)}_{-Iw'}\right)$$

$$= \underset{\in V_+}{(w, Iw)} + \underset{\in V_-}{(w', -Iw')} \qquad w' = \frac{u+Iv}{2}$$

$$w = \frac{u-Iv}{2} \qquad \Rightarrow \mathfrak{g} \oplus \mathfrak{g} = V_+ \oplus V_-$$

$$L \cdot (w, Iw) = J(Iw, -w) = (w, Iw)$$

($V_\pm$ are ideals as explained in Ziller)

$$L(w, -Iw) = J(Iw, w) = -(w, -Iw)$$

$f_- \; \mathfrak{g} \to V_-$ is a Lie algebra isomorphism since

$$f_-([u, v]) = \left(\tfrac{1}{2}[u, v], \frac{I([u, v])}{2}\right)$$

$$[f_-(u), f_-(v)] = \left[\tfrac{1}{2}(u, -Iu), \tfrac{1}{2}(v, -Iv)\right]$$

$$\overset{(*)}{=} \left(\tfrac{1}{4}([u, v] + [u, v]), \tfrac{1}{4}\left([u, -Iv] + \underbrace{[-Iu, v]}\right)\right)$$

$$\qquad\qquad\qquad\qquad \overset{\shortparallel}{-\tfrac{1}{2} I([u, v])}$$

Moreover

$$f_-(Iu) = \tfrac{1}{2}(Iu, -I(Iu)) = \tfrac{1}{2}(Iu, u)$$

$$= J \cdot f_-(u)$$

$\Rightarrow f_-$ is a complex Lie algebra isomorphism

Similarly $f_+ : \bar{\mathfrak{g}} \longrightarrow V_+$ is an Lie algebra isomorphism.

3 Realification $\quad$ If $\mathfrak{g}$ is a complex Lie algebra

We may forget the multiplication of i to treat it as a real Lie algebra. $(\mathfrak{g})$

If $e_1, \dots e_r$ are the basis of $\mathfrak{g}$

$\{e_1 \dots e_r, ie_1, \dots ie_r\}$ forms a basis of $(\mathfrak{g})$

$[e_k, ie_j] = i[e_k, e_j] = [ie_k, e_j]$.

$[ie_k, ie_j] = -[e_k, e_j]$

Now if $\mathfrak{g}_c$ is the complexification of $\mathfrak{g}$ & $(\mathfrak{g}_c)$ is its realification

$(\mathfrak{g}_c) = \mathfrak{g} \oplus \mathfrak{g}$ with the Lie algebra bracket

↑ again as a vector space.

$$[(u,v), (x,y)] = ([u, x] - [v, y], [u, y] + [v, x])$$

$\underline{u+iv}$ $u, v, x, y \in \mathfrak{g}$

$I_+ \mathfrak{g} \longrightarrow (\mathfrak{g}_c)$

$u \longrightarrow (u, 0)$

$$I_+([u, v]) = ([u, v], 0)$$

$$\| $$

$$[I_+(u), I_+(v)] = [(u,0), (v,0)]$$

$$= ([u, v], 0)$$

$\psi : (u, v) \longrightarrow (u, -v)$

ψ is an isomorphism

$$\psi([(u, v), (x, y)]) = [\psi(u, v), \psi(x, y)]$$

$$\|$$

$$([u, x] - [v, y], -([u, y] + [v, x])) \Leftarrow [(u, -v), (x, -y)]$$

Namely $\psi : (\mathfrak{g}_c) \to (\mathfrak{g}_c)$ is a Lie algebra isomorphism

$I_+(\mathfrak{g}) \subset (\mathfrak{g}_c)$ is the one containing all z $\quad \psi(z) = z$.

Now we prove the following.

Let R be the maximum solvable ideal of $\mathfrak{g}$, a real Lie-algebra then R_c is the maximum solvable ideal of $\mathfrak{g}_c$. In fact $(\mathrm{Rad}(\mathfrak{g}))_c = \mathrm{Rad}(\mathfrak{g}_c)$

Pf: Let T be the maximum solvable ideal of $\mathfrak{g}_c$

$$\psi(T) := \{ u - iv \mid u + iv \in T \}$$

(a) $\psi(T)$ an ideal.

$$\begin{aligned}
&[u-iv,\ x+iy] \\
&= ([u,x]+[v,y]) + i([u,y]-[v,x]) \\
&= [u-iv,\ x-i(-y)] \\
&= [u,x]-[v,(-y)] - i([u,(-y)]+[v,x]) \\
&= \psi([u+iv,\ x+i(-y)]) \\
&\qquad\qquad\qquad\quad \in T
\end{aligned}$$

Namely $[u-iv, x+iy] \in \psi(T)$ Hence it is an ideal.

(b) Moreover if T is solvable $\Rightarrow$ $\psi(T)$ is solvable.

$$[u+iv,\ x+iy] = ([u\ x]-[v,y] + i([u,y]+[v,x]))$$

$$[u-iv,\ x-iy] = ([u\ x]-[v,y] - i([u,y]+[v,x]))$$

$$\Longrightarrow \quad [\psi(T), \psi(T)] = \psi([T,T]) \qquad (\psi(T))_1 = \psi(T_1)$$

Hence if T is solvable $\Rightarrow \psi(T)$ is.

By the maximality of $T \Rightarrow$ $\quad T = \psi(T)$.

$\Rightarrow$ $T = S_C$ for some S [This need some calculations].

It can be shown that S is an ideal in $\mathfrak{g}$. T is solvable iff S is solvable $\Rightarrow$ $S \subset R$.

But $R \subset S$ as well since R_c is a solvable ideal of $\mathfrak{g}_c$

$R_c \subset T = S_c$ This shows $T = R_c$

<u>Thm</u>: $\mathfrak{g}$ is semi-simple iff $\mathfrak{g}_c$ is semi-simple. $R(\mathfrak{g})=0 \Leftrightarrow R(\mathfrak{g}_c)=0$

[In terms of the 'No maximal solvable ideals.' $Rad(\mathfrak{g})$]

④ The Killing form.

Let B be the Killing form of $\mathfrak{g}$, a real Lie algebra

$B(\cdot,\cdot)$ extends bilinearly to $\mathfrak{g}_c$

Namely $B(z,w) = B(x+iy, u+iv) \doteq [B(x,u) - B(y,v)] + i(B(x,v) + B(y,u))$

$z = x+iy$, $w = u+iv$ $\in \mathfrak{g}_c$

we denote it by B_c

<u>Claim</u>: $Rad(B)_c = Rad(B_c)$.

<u>Pf</u>: If $z \in Rad(B_c) := \{ z \mid B_c(z,w) = 0 \ \forall w \in \mathfrak{g}_c \}$

then $B_c(x+iy, u) = 0 \quad \forall u \in \mathfrak{g}$

$\Rightarrow$ $B(x,u) + i B(y,u) = 0 \Rightarrow y, x \in Rad(B)$

$\Rightarrow x+iy \in (Rad(B))_c$

Namely $Rad(B_c) \subset (Rad(B))_c$.

On the other hand $\forall u+iv \in Rad(B)_c$

$B(u+iv, x+iy) = 0$ clearly $\Rightarrow$ $u+iv \in Rad(B_c)$

Combining Thm & the above we have

Corollary: Cartan's criterion holds for $\mathfrak{g}$. — real Lie algebra.

Namely $\mathfrak{g}$ is semi-simple iff $Rad(B) = \{0\}$.

pf. If $Rad(B)=\{0\} \Rightarrow Rad(B_c)=\{0\} \Rightarrow \mathfrak{g}_c$ is semi-simple
$\Rightarrow \mathfrak{g}$ is

If $\mathfrak{g}$ is semi-simple $\Rightarrow \mathfrak{g}_c$ is $\Rightarrow Rad(B_c)=\{0\} \Rightarrow Rad(B)=\{0\}$.

□

Example: on $\mathfrak{sl}(n, \mathbb{C})$.

$$B(X, Y) = 2n\, tr(XY)$$

Even on $\mathfrak{sl}(n, \mathbb{R})$ B is Not positive/negative definite.

But it is non-degenerate!

$SL(n, \mathbb{C})$ - complex Lie group with Lie algebra $\mathfrak{sl}(n, \mathbb{C})$

$$\underbrace{SU(n) \times \mathbb{R}^{n^2-1}}_{} \overset{top}{\cong}$$
which is NoT compact

Similarly $SL(n, \mathbb{R})$ is NoT compact since $\overset{top}{\cong} SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}-1}$

On $\mathfrak{sl}(n, \mathbb{R})$: B is non-degenerate. But $-B$ is NoT positive definite. Check on $\mathfrak{sl}(2, \mathbb{R})$.

$\mathfrak{sl}(n, \mathbb{C})$ is semi-simple

$\mathfrak{su}(n)$ $\quad B(X, Y) = 2n\, tr(X, Y) \Rightarrow B(X,X) = -2n\, tr(X, X^*) < 0$

$X + X^* = 0 \quad \Rightarrow \mathfrak{su}(n)$ is simple.

$\mathfrak{u}(n)$ complexify into $\mathfrak{gl}(n, \mathbb{C})$ which is not semi-simple — $X = \sqrt{-1}\, id \in \mathfrak{u}(n)$

$ad_X \equiv 0$