(1) $\exists$ a very nice book "Convex Figures" Yaglom.
used at Moscow State.

Convex functions are very much related to the convex bodies $U_f := \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} \mid z \geq f(x) \}$

is a convex body $(1-t)(x_1, z_1) + t(x_2, z_2)$

$\in U_f$ since

$(1-t)z_1 + t z_2 \geq (1-t) f(x_1) + t f(x_2) \geq f((1-t)x_1 + t x_2)$.

"$f$ convex" $\Longrightarrow$ $f$ is locally Lipschitz $\Longleftrightarrow$ $f \in W^{1,\infty}_{loc}$

Hence $Df$ — the distributional derivative is $L^\infty$.

<u>Theorem 6.9</u>: $f: \mathbb{R}^n \to \mathbb{R}$ is convex.

Then
$$\left| f(y) - f(x) - Df(x)(y-x) - \frac{1}{2}(y-x)^T \cdot D^2 f(x) \cdot (y-x) \right|$$
$$= o(|y-x|^2) \quad \text{as } y \to x$$

a.e $x \in \mathbb{R}^n$.

Before we prove the thm we need to make sense of $D^2 f(x)$.

(A). This is NOT the distributional $\overset{\text{2nd}}{\widehat{\text{derivative}}}$ of $f$!

Since $f$ is Lipschitz $\Rightarrow$ $f_i = \partial_i f \in L^\infty_{loc}$

$\Longrightarrow f_i \in L^1_{loc}$.

Hence $f_{ij}(x)$ needs some deliberations.

(B) $f_1, \dots f_n \in BV_{loc}(\mathbb{R}^n)$.

To obtain this observe that:

$$f^\varepsilon = j_\varepsilon * f \quad \text{satisfies} \quad D^2 f^\varepsilon \geq 0.$$

$$\Rightarrow \sum_{i,j=1}^{n} \int f^\varepsilon \phi_{x_i x_j} V^i V^j = \int \phi \sum f^\varepsilon_{x_i x_j} V^i V^j$$

$$\downarrow \; \varepsilon \to 0$$

$$\Rightarrow L(\phi) := \int f \phi_{x_i x_j} V^i V^j \geq 0 \quad \forall \phi \geq 0, \; \phi \in \mathcal{D}(\mathbb{R}^3).$$

By RM $\Rightarrow$ $\forall V \in S^{n-1}$ $\Rightarrow$

$$L_V(\phi) = \int \phi \, d\mu^V \qquad \mu^V \text{ is a Radon measure!}$$

$$\Rightarrow \mu^{ii} := \mu^{e_i} \qquad V = \frac{e_i + e_j}{\sqrt{2}}$$

$$\Rightarrow \sum \phi_{kl} V^k V^l = \frac{1}{2} \sum \left( \phi_{x_i x_i} + 2\phi_{x_i x_j} + \phi_{x_j x_j} \right)$$

$$\Rightarrow \int f \phi_{x_i x_j} = \int f \left( \phi_{VV} - \tfrac{1}{2}\phi_{ii} - \tfrac{1}{2}\phi_{jj} \right)$$

$$\Rightarrow \mu^{ij} = \mu^V - \tfrac{1}{2}\mu^{ii} - \tfrac{1}{2}\mu^{jj}$$

Clearly $(\mu^{ij})$ is a $\geq 0$ matrix of measures.

Namely $\mu^V = \sum (\mu^{ij}) V^i V^j$ is a positive measure.

Now we may make sense of the derivative of $f_k$

$$\int f_k \, \mathrm{div}(\phi) = \int f_k \sum \phi^i_{x_i} = -\int f \sum \phi^i_{x_i x_k}$$

$$= - \int \phi^i \, d\mu^{ik}$$

Hence $L_k(\phi) \leq C \sum_i |\mu|^{ik}(K)$ if $\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$.
spt $\phi \subset K$ $|\phi| \leq 1$

$\Rightarrow$ $f_k$ is a BV function
by RM - essentially the definition!

(c) $\mu^{ij}_{ac} = f_{ij} \, d\mathcal{L}^n$ $\mu^{ij}_s$ — the singular part of $\mu^{ij}$ w.r.t $d\mathcal{L}^n$ - Lebesgue.

Now $D_{ij} f = f_{ij}$.

Namely in Alexandrov's theorem $D^2 f(x)$ is the regular part of the distributional $\underbrace{\text{derivative}}_{\text{2nd}}$ of $f$.

$[D^2 f]_d$ - means $(\mu^{ij})$ - the Radon measure

$[D^2 f] = (f_{ij})$ $[D^2 f]_s = \mu^{ij}_s$ — the singular part of $[D^2 f]_d$.

② Proof of the Thm 6.9 :

(i) $\forall$ a.e $x$ $\displaystyle\lim_{r\to 0} \fint_{B(x,r)} |Df(y) - Df(x)| = 0$.

(ii) $\displaystyle\lim_{r\to 0} \fint_{B(x,r)} |D^2 f(y) - D^2 f(x)| = 0$

(iii) $\displaystyle\lim_{r\to 0} \frac{[D^2 f]_s (B(x,r))}{|B(x,r)|} = 0$

(ii) & (iii) follows from Lebesgue differentiation theorem.

[Thm 3.22 of Folland].

Below we shall show that if $x$ is a point where (i), (ii), (iii) hold, then Theorem hold at $x$.

Also $f^\varepsilon(x) \to f(x)$ & $Df^\varepsilon(x) \to f(x)$.

The tricky part is to get $o(r^2)$ estimate on

$$f(y) - f(x) - Df(x)\cdot(y-x) - \frac{1}{2}(y-x)^T D^2 f(x)(y-x).$$

[Fix $\varepsilon$, it is obviously true for $f^\varepsilon$]

W.L.G assume $x=0$.

Consider $H^\varepsilon(y) = f^\varepsilon(y) - f^\varepsilon(0) - Df^\varepsilon(0)\cdot y - \frac{1}{2} y^T D^2 f(0) y$

We also denote

$$h(y) := f(y) - f(0) - Df(0)\cdot y - \frac{1}{2} y^T \cdot D^2 f(0)\cdot y$$

The goal is to show $|h(y)| = o(|y|^2)$. $H^\varepsilon(y) \longrightarrow h(y)$ a.e.

Taylor expansion of $f^\varepsilon(y) \Longrightarrow$

$$f^\varepsilon(y) = f^\varepsilon(0) + \int_0^1 \frac{d}{dt} f^\varepsilon(ty) \Longrightarrow f^\varepsilon(y) - f^\varepsilon(0) = \int_0^1 Df^\varepsilon(ty)\cdot y\, dt$$

$$= \int_0^1 Df^\varepsilon((1-t)y)\cdot y\, dt = Df^\varepsilon(0)\cdot y + \int_0^1 y^T \cdot D^2 f^\varepsilon_{(1-t)y}\, y\, t$$

$$= Df^\varepsilon(0)\, y + \int_0^1 (1-s)\, y^T D^2 f^\varepsilon(sy)\cdot y\, ds.$$

This shows

$$H^\varepsilon(y) = \int_0^1 \left[(1-s)\left(y^T D^2 f^\varepsilon(sy)\, y - y^T D^2 f(0)\, y\right)\right] ds$$

$\Big\{$ $D^2 f^\varepsilon(sy) \to D^2 f(0)$ only in averge sense. by Lebesgue differentiation.

Claim: $\forall \phi \in C_c^\infty(B(r))$

$$\fint h(y)\cdot \phi(y) = o(r^2) \quad \text{as } r \to 0 \qquad (*)$$

Proof of $(*)$

$$\fint_{B(r)} \phi(y)\, H^\varepsilon(y) = \int_0^1 (1-s)\left(\fint_{B(r)} \phi(y)\, y^T \cdot \Big[D^2 f^\varepsilon(\underset{\substack{\parallel \\ z}}{sy}) - D^2 f(0)\Big]\, y\, dy\right) ds$$

$$= \int_0^1 \frac{(1-s)}{s^2}\left(\fint_{B(rs)} \phi\left(\frac{z}{s}\right) z^T \left[D^2 f^\varepsilon(z) - D^2 f(0)\right] z\, dz\right) ds$$

Let

$$g_\varepsilon(s) := \underbrace{\int_{B(rs)} \phi\left(\frac{z}{s}\right) z^T D^2 f^\varepsilon(z)\cdot z\, dy}$$

$$= \int_0^1 \frac{(1-s)}{s^2} \frac{1}{\alpha(n) r^n s^n} \left[ g_\varepsilon(s) - \int_{B(rs)} \phi\left(\tfrac{z}{s}\right) z^T D^2 f(0)\, z \right] ds$$

$$g_\varepsilon(s) = \int_{B(rs)} f^\varepsilon(z) \left( \sum \left[ \phi\left(\tfrac{z}{s}\right) z_i z_j \right]_{z_i z_j} \right) dz$$

$$\Big\downarrow \quad \text{as } \varepsilon \to 0$$

$$\int_{B(rs)} f(z) \sum \left[ \phi\left(\tfrac{z}{s}\right) z_i z_j \right]_{z_i z_j} dz$$

$$= \int_{B(rs)} \phi\left(\tfrac{z}{s}\right) z_i z_j \, d\mu^{ij}$$

$$= \int_{B(rs)} \phi\left(\tfrac{z}{s}\right) z_i z_j D^2_{ij} f + \phi\left(\tfrac{z}{s}\right) z_i z_j \underbrace{\left( [D^2 f]_s \right)_{ij}}_{d\mu_s^{ij}}$$

To apply DCT on $g_\varepsilon(s)$ we need an estimate.

Sub-claim: $\dfrac{|g_\varepsilon(s)|}{s^{n+2}} \le C$

If holds, $\Longrightarrow$

$$\int_{B(r)} \phi(y) H^\varepsilon(y) \longrightarrow \int_0^1 \frac{(1-s)}{s^2} \frac{1}{r^n s^n \alpha(n)} \int_{B(rs)} \phi\left(\tfrac{z}{s}\right) z_i z_j \left[ D^2_{ij} f(z) - D^2_{ij} f(0) \right] + \phi\left(\tfrac{z}{s}\right) z_i z_j \, d\mu_s^{ij}$$

$$\Big\downarrow$$

$$\int \phi(y) h(y)$$

Hence $\displaystyle \fint_{B(r)} \phi(y)\,h(y) \leq C\,r^2 \int_0^1 \Big( \underbrace{\fint_{B(rs)} |D^2 f(z) - D^2 f(0)|}_{=o(1)} + \underbrace{\frac{|[D^2 f]_s|(B(rs))}{(rs)^n}} \Big) ds$

$= o(r^2)$ as $r \to 0$.

This proves $(*)$.

③ Proof of subclaim & finishing the argument.

Ⓐ $\displaystyle \fint_{B(r)} |h(y)|\,dy = o(r^2)$

Namely $|h(y)| = o(|y|^2)$ in average sense.

It suffices to get estimate for $\phi$ on $\displaystyle \fint_{B(r)} \phi\,h$    $\phi \in C_c^\infty(B(r))$, $|\phi| \leq 1$

since one may approximate $\operatorname{sgn}(h)$ by $\phi$

Ⓑ $\exists\, C$

$$\sup_{B(\frac{r}{2})} |Dh| \leq \frac{C}{r} \int_{B(r)} |h|\,dy + Cr.$$

Recall. $h = f(y) - \underbrace{f(0) -}\, Df(0)\cdot y - \frac{1}{2} y^T \cdot D^2 f(0)\, y$

Let $\Lambda = |D^2 f(0)| \Rightarrow$    $h + \frac{\Lambda}{2}|y|^2$ is convex.

Since $-f(0) - Df(0)\cdot y - \frac{1}{2} y^T D^2 f(0)\, y + \frac{\Lambda}{2}|y|^2$ is convex.

Now by Thm 6.7

$$\sup_{B(\frac{r}{2})} \left| D\left(h + \frac{\Lambda}{2}|y|^2\right) \right| \le \frac{C}{r}\left( \fint_{B(r)} |h| + \frac{\Lambda}{2}|y|^2 \right)$$

$\le C r$

this $C$

(c) $\sup_{B(\frac{r}{2})} |h| = o(r^2)$ as $r \to 0$

This proves the theorem. $\forall \varepsilon > 0$

pick $\eta \in (0,1)$, $\eta^{\frac{1}{n}} < \frac{1}{2}$, $C(2\eta)^{\frac{1}{n}} = \varepsilon$.

We shall show

$$|h(y)| \le 2\varepsilon r^2 \quad \text{if} \quad r \le r_0 := r_0(\varepsilon, \eta).$$

Step 1. $\frac{1}{\varepsilon} \fint_{B(r)} |h| < r^2 \eta\, r^n \alpha(n)$ if $r \le r_0$

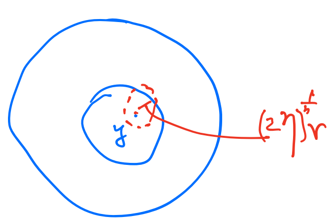
by (B).

Then $\left| \{ z \in B(r) \mid |h|(z) \ge \varepsilon r^2 \} \right| \le \eta\, |B(r)|$

Since

$$\underbrace{\left| \{ z \in B(r) \mid |h|(z) \ge \varepsilon r^2 \} \right|}_{S} \cancel{\varepsilon r^2} \le \int_{B(r)} |h| \le \cancel{\varepsilon}\, \eta\, \cancel{r^2}\, |B(r)|$$

Now $\forall y \in B(\frac{r}{2})$



If $B(y,(2\eta)^{\frac{1}{n}}r) \cap S^c = \phi \Rightarrow B(y,(2\eta)^{\frac{1}{n}}r) \subset S$

$\Rightarrow |S| \geq 2\alpha(n)\eta r^n$

Hence $\Rightarrow \exists z$, $|h(z)| < \varepsilon r^2$ $z \in B(y, (2\eta)^{\frac{1}{n}}r)$

But $|h(y)| \leq |h(z)| + \underbrace{|h(z)-h(y)|}$

$\leq \varepsilon r^2 + C(2\eta)^{\frac{1}{n}} r^2 \leq 2\varepsilon r^2$

D Proof of the Sub-claim.

We only need to establish it for $\varepsilon \ll 1$ & $r$ small.

$$\frac{|g_\varepsilon(s)|}{s^{n+2}} \leq \frac{r^2}{s^n}\int_{B(rs)} |D^2 f^\varepsilon(z)|dz$$

$$= \frac{r^2}{s^n}\int_{B(rs)} \left|\int D^2 j_\varepsilon(z-y) f(y)\, dy\right| dz$$

$$= \frac{r^2}{s^n}\int_{B(rs)} \left|\int j_\varepsilon(z-y)\, \underbrace{d_y D^2_d f}_{\substack{\uparrow \\ \text{measure}}}\right| dz$$

$$\leq \frac{C r^3}{\varepsilon^n s^n}\int_{B(rs+\varepsilon)} \left(\int_{B(rs)\cap B(y,\varepsilon)} dz\right) d_y D^2_d f$$

$$\leq C r^2 \frac{\min((rs)^n, \varepsilon^n)}{\varepsilon^n s^n} \cdot (D^2_d f)\underbrace{(B(rs+\varepsilon))}_{(\leq C(rs+\varepsilon)^n)}$$

if $rs \leq \varepsilon$: $\leq C\frac{(rs)^n \varepsilon^n}{\varepsilon^n s^n}$

if $rs \geq \varepsilon$: $\leq C\frac{\varepsilon^n (rs)^n}{\varepsilon^n s^n}$ as well.

$$\leq C r^{n+2}$$

using $r$ & $\varepsilon$ small.

$$D^2_d f = D^2 f + [D^2 f]_s$$