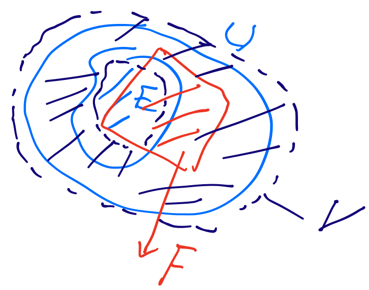
1 Regularity of Radon measures.

Prop 7.5. μ Radon E is σ-finite, then μ is inner regular on E.



Pf: $\forall \varepsilon > 0 \ \exists U \quad \mu(U) \leq \mu(E) + \varepsilon/2$

$\mu(E) < \infty \qquad \exists F \subset U$ compact

$\mu(F) \geq \mu(U) - \varepsilon/2$

Consider $U \setminus E \quad \exists \underset{U\setminus E}{\overset{V}{\cup}}$ open $\quad \mu(V) \leq \mu(U\setminus E) + \frac{\varepsilon}{2}$

$(U \cap E^c) \subset V \Rightarrow V^c \subset \underset{U^c \cup E}{\overset{(U\cap E^c)^c}{\shortparallel}}$

$\Rightarrow \quad F \setminus V$ is compact $\qquad \underset{F\cap V^c}{\overset{F\setminus V}{\shortparallel}} \subset E$

$\Rightarrow \begin{array}{l} F \subset U \\ F \cap V^c = F \cap E \subset E \end{array}$

$\mu(F\setminus V) \geq \mu(F) - \mu(V)$

$\geq \mu(U) - \varepsilon/2 - \mu(U\setminus E) - \varepsilon/2 \ = \ \mu(E) - \varepsilon.$

Now if $\mu(E) = \infty$. But E is σ-finite $\Rightarrow E = \bigcup E_j$.

$E_j \nearrow \qquad \mu(E_j) < +\infty$

$\Rightarrow \forall N, \exists j_0 \quad \mu(E_{j_0}) \geq N+1 \quad$ Now, $\exists \ K \subset E_{j_0} \quad \mu(K) \geq N.$

Corollary: If X is σ-compact, then Radon measure is regular (on all Borel sets).

2 Regularity of Borel measures.

Thm 7.8. μ is Borel. Every U is σ-compact

Then if $\mu(K)<\infty$ $\forall$ K compact $\Rightarrow$ μ is regular.

Corollary: If μ is Borel on $\mathbb{R}^n$. $\mu(K)<+\infty$

$\Longrightarrow$ μ is regular.

On page 99 of Folland, the definition of regular measure has an additional assumption, which is NOT needed.

[Lemma 2.43. $U \subset \mathbb{R}^n$ open U is σ-compact]

Pf: $I(f) = \int_X f\,d\mu \quad \forall\ f \in C_c(X)$.

$\Longrightarrow \exists\ \nu$ - a Radon measure

$$I(f) = \int_X f\,d\nu \qquad \forall\ f \in C_c(X).$$

Step 1: $\forall\ U$ open $\mu(U) = \nu(U)$.

$U = \bigcup K_j$. $\quad \exists\ f_j \quad f_j \prec U \quad f_j \leq f_{j+1}$

$K_j \uparrow$

f_{j+1} is constructed such that $K_j \cup \operatorname{supp} f_j \prec f_{j+1} \prec U$

Namely $f_{j+1} = 1$ on $\operatorname{supp} f_j$

$f_j \longrightarrow \chi_U$

$$\lim_{j\to\infty} \int_X f_j\,d\mu = \int \chi_U = \mu(U).$$

But $I(f_j)$ Satisfies $I(f_j) \geq \nu(K_j)$

$$\nu(U) = \lim_{j\to\infty} \nu(K_j)$$

Step 2: μ is outer regular.

Lemma: $\forall E$ Borel, $\varepsilon > 0$ $\exists$ V open $V \supset E$ & F closed $F \subset E$

Such that $\nu(V \setminus F) < \varepsilon$.

Assume Lemma $\nu(V\setminus F) = \mu(V\setminus F) < \varepsilon \Rightarrow \mu(V \setminus E) < \varepsilon$

$\Rightarrow \mu(E) \geq \mu(V) - \varepsilon \quad \forall \varepsilon$

$\Rightarrow$ μ is outer regular on E

Step 3. μ is inner regular.

$\mu(F) \geq \mu(E) - \varepsilon$ (by Lemma)

$$F = \bigcup_{j=1}^{\infty} \underbrace{K_j \cap F}_{\tilde{K}_j} \qquad \left(\bigcup K_j = V\right) \quad K_j \uparrow$$

$\Rightarrow \lim_{j\to\infty} \tilde{K}_j = F \Rightarrow \mu(F) = \lim_{j\to\infty} \mu(\tilde{K}_j) \geq \mu(E) - \varepsilon$

$\Rightarrow \mu(\tilde{K}_j) \geq \mu(E) - 2\varepsilon$.

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Now proof of the Lemma: (Prop 7.7).

ν is a Radon measure

$\forall E$ Borel, if $E \subset U$ ($U = X$ for example).

Since $U = \bigcup K_j \Rightarrow E \subset \bigcup K_j$

$\Rightarrow \quad E = \bigcup_{j=1}^{\infty} E \cap K_j \qquad \nu(E \cap K_j) \leq \nu(K_j) < \infty$

$\Rightarrow$ Any E is σ-finite.

Now write $E = \bigcup E_j$

Since $\forall E_j \quad \exists U_j \supset E_j \quad \nu(E_j) > \nu(U_j) - \varepsilon/2^j$

$\Rightarrow \quad U = \bigcup U_j \quad \Rightarrow \qquad \nu(U_j \setminus E_j) \leq \frac{\varepsilon}{2^j}$

$\nu(U \setminus E)$

$$U \setminus E = \left(\bigcup U_j\right) \cap (E^c)$$

$$= \bigcup_j (U_j \cap E^c) \qquad U_j \cap E^c \subset U_j \cap E_j^c$$

$\Rightarrow \nu(U \setminus E) \leq \sum \nu(U_j \setminus E_j) \leq \varepsilon.$

Apply it to $E^c \Rightarrow \exists \ U'$ such that $E^c \subset U'$

$$\nu(U' \setminus E^c) \leq \varepsilon$$

Now consider $F = (U')^c \Rightarrow F \subset E \subset U$

$$\nu(U \setminus F) \leq \nu(U \setminus E) + \nu(E \setminus F)$$

$$\leq \varepsilon + \nu(E \setminus F)$$

$$E \cap F^c = E \cap U' = U' \cap (E^c)^c$$

$\Rightarrow \nu(E \setminus F) = \nu(U' \setminus E^c) \leq \varepsilon$

$\Big\} \Rightarrow \nu(U \setminus F) \leq 2\varepsilon.$