Riesz Representation:

X LCH. I: $C_c(X) \longrightarrow \mathbb{R}$ a positive linear functional.

Namely I(f)$\geq$ **0** for any f$\geq$0. Then $\exists$ σ-algebra $\mathcal{M}$ $\mathcal{B} \subset \mathcal{M}$.

& μ- measure satisfies:

(i) $I(f) = \int_X f \, d\mu \quad \forall f \in C_c(X)$.

(ii) μ is Radon.

(iii) $\mu(U) = \sup \{ I(f), \ f \prec U \}, \ \forall U$ open

(iv) $\mu(K) = \inf \{ I(f), \ f \geq \chi_K \}. \ \forall K$ compact.

Moreover, such μ is unique.

<u>Notation</u>: $K \prec f \prec U$. $f \in C_c(X, [0,1])$

$f \succ K \iff f \in C_c(X) \quad f \geq \chi_K$

$f \prec U \iff \operatorname{supp} f \subset U$.

Uryshon Lemma for LCH $\Rightarrow$. $\forall K \subset U$ K compact U open

$\exists f \quad K \prec f \prec U$.

<u>Uniqueness</u>: If μ_1 μ_2 are two Radon measures satisfy Thm.

Given K.

$\forall \varepsilon > 0$, $\exists U$ $K \subset U$ & $\mu_i(U) \leq \mu_i(K) + \varepsilon$.

$$\mu_1(K) = \int_K d\mu_1 = \int_X \chi_K \, d\mu_1 \leq \int_X f \, d\mu_1 = I(f)$$

$$= \int_X f \, d\mu_2 \leq \int_X \chi_U \, d\mu_2 \leq \mu_2(U) \leq \mu_2(K) + \varepsilon$$

Similarly $\mu_2(K) \leq \mu_1(K) + \varepsilon$. $\Longrightarrow \mu_1(K) = \mu_2(K)$.

Construction of μ: (Existence).

$$\mu(U) := \sup\{I(f),\ f \prec U\}$$
$$\mu^*(E) := \inf\{\mu(U),\ \forall E \subset U\}$$

Steps (i) μ^* is an outer measure

(ii) U is measurable

(iii) $\mu(K) = \inf\{I(f),\ f \succ K\}$ (7.4)

& $\mu(K) < +\infty$

(iv) μ is Radon

(v) $I(f) = \int_X f\, d\mu$

We shall do them (i) — (v).

Pf: (i) Let $\tilde{\mu}^*(E) := \inf\left\{ \sum_{i=1}^{\infty} \mu(U_i) \,\middle|\, E \subset \bigcup_{i=1}^{\infty} U_i \right\}$

By Prop 1.10. $\tilde{\mu}^*(E)$ is an outer measure.

Now we claim $\tilde{\mu}^*(E) = \mu^*(E)$

It follows from $\mu(U) \leq \sum \mu(U_i)$ (*) if $U = \bigcup_i U_i$.

By definition, $\tilde{\mu}^*(E) \leq \mu^*(E)$.

On the other hand. $\forall\ U \supset E$ & $E \subset \bigcup_{i=1}^{\infty} U_i = U$

$$\mu(U) \leq \sum \mu(U_i)$$

$$\Rightarrow \quad \mu^*(E) \leq \tilde{\mu}^*(E)$$

Now we prove (*).

$\forall f \prec U \Rightarrow K = \operatorname{supp} f$ is compact

$\exists \quad \sum_{i=1}^{n} g_i = 1 \qquad K \subset \bigcup_{i=1}^{n} U_i$

$f = \sum f g_i$

$I(f) = \sum_{i=1}^{n} I(f g_i) \leq \sum_{i=1}^{n} \mu(U_i) \leq \sum_{i=1}^{\infty} \mu(U_i)$

$\Rightarrow$ By $\mu(U) := \sup \{ I(f) \mid f \prec U \}$

$\mu(U) \leq \sum_{i=1}^{\infty} \mu(U_i)$.

(ii) $\quad \mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$. $\quad \forall U$ open. $\quad$ (*2)

Only need to check for $\mu^*(E) < +\infty$

Reduction only need to check for $E = U_1$ open.

Proof: $\uparrow$ $\forall \varepsilon > 0 \ \exists U_1$ open $\quad \mu(U_1) - \varepsilon \leq \mu^*(E) \quad E \subset U_1$

Then if we have

$$\mu(U_1) = \mu(U_1 \cap U) + \mu(U_1 \setminus U)$$
$$(\geq)$$

Then $\quad \mu^*(E \cap U) + \mu^*(E \setminus U) \leq \mu(U_1 \cap U) + \mu(U_1 \setminus U)$

$$\leq \mu(U_1) \leq \mu^*(E) + \varepsilon$$

$\Rightarrow$ (ii) holds.

Now we check (*)2 for open subsets.

$\forall \varepsilon > 0 \ \exists f \prec E \cap U$

$\mu(E \cap U) \leq I(f) + \varepsilon$

$K = \operatorname{Supp} f \subset E \cap U$ Consider $E \setminus K$

$\exists\, g \prec E \setminus K$ $\quad I(g) + \varepsilon \geq \mu(E \setminus K)$

$\operatorname{supp} g$ $\qquad f + g \prec E$

$$\Rightarrow \quad I(f+g) + 2\varepsilon \geq \mu(E \cap U) + \mu(E \setminus K)$$

$$\geq \mu(E \cap U) + \mu(E \setminus U) \qquad E \setminus U \subset E \setminus K$$

$$\Rightarrow \quad \mu(E) + 2\varepsilon \geq \mu(E \cap U) + \mu(E \setminus U). \quad \Rightarrow \quad (*)\ 2.$$

(iii) By definition $\mu(K) = \inf \{ \mu(U),\ K \subset U \}$.

$\forall \quad K \subset U \quad \exists\, f \quad K \prec f \prec U$

$$\Rightarrow \quad I(f) \leq \mu(U) \quad \Rightarrow$$

$$\inf \{ I(f),\ f \succ K \} \leq \inf_{K \subset U} \{ \mu(U) \}$$

Hence RHS of (7.4) $\leq \mu(K)$.

For the other direction, $\forall\, f \succ K$

$U_\varepsilon := \{ f > 1 - \varepsilon \}$ open $\quad$ & $K \subset U_\varepsilon$

$$\forall \quad g \prec U_\varepsilon \quad \Longrightarrow \quad \frac{1}{1-\varepsilon} f \geq g$$

$$\Rightarrow \quad \underset{\frac{1}{1-\varepsilon} I(f)}{\underset{\|}{I\left(\frac{1}{1-\varepsilon} f\right)}} \geq I(g) \quad \Longrightarrow \quad \mu(U_\varepsilon) \leq \frac{1}{1-\varepsilon} I(f)$$

$$\Rightarrow \quad \mu(K) \leq \frac{1}{1-\varepsilon} I(f) \quad \Longrightarrow \quad \mu(K) \leq \text{RHS of (7.4)}$$

Now $\mu(K) \leq I(f)$ for $f \succ K$

$$\Rightarrow \mu(K) < +\infty$$

(iv) Namely μ is outer regular on Borel ($\checkmark$ by definition)

& inner regular on open set.

Namely $\mu(U) = \sup \{ \mu(K), \quad K \subset U \}$.

$\Leftrightarrow \ \forall \alpha < \mu(U). \quad \exists K \quad \mu(K) > \alpha$

By definition of $\mu(U)$ $\exists \quad f \prec U \quad I(f) > \alpha$

Let $K = \operatorname{supp} f$

$K \subset\subset U \quad \exists g \qquad K \prec g \prec U$

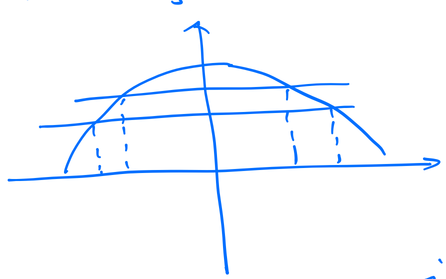
$\Rightarrow \qquad \mu(K) \geq I(g) - \varepsilon \qquad$ (by (iii))

But $\qquad g \geq f \quad \Rightarrow \quad I(g) \geq I(f)$

$\mu(K) \geq I(f) - \varepsilon. \qquad \forall \varepsilon \quad \Rightarrow \quad \mu(K) \geq I(f) > \alpha$

Finally (v): $\qquad f \in C_c(X), \quad 0 \leq f \leq 1$

$\forall N$ big natural #.



$K_j \subset K_{j-1}$

each one of them is compact

$K_0 = \operatorname{supp} f$

$$K_j := \{ x : \ f(x) \geq \frac{j}{N} \} \qquad N \geq j \geq 1$$

$$f_j := \begin{cases} f(x) - \frac{j-1}{N} & x \in K_{j-1} \setminus K_j \\ \frac{1}{N} & x \in K_j \\ 0 & x \notin K_{j-1} \end{cases}$$

Claim: ① $\frac{1}{N}\mu(K_j) \leq \int f_j \, d\mu \leq \frac{1}{N}\mu(K_{j-1})$ ✓ Easy

② $\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$

③ $f = \sum_{j=1}^{N} f_j$ geometrically clear.

For ② $N f_j \succ K_j \Rightarrow I(f_j) \geq \frac{1}{N}\mu(K_j)$

For any open subset $U \supset K_{j-1}$ & $U \succ g \succ K_{j-1}$

$\Rightarrow g \geq N f_j \Rightarrow I(g) \geq I(N f_j)$

$\Rightarrow \mu(U) \geq N I(f_j) \Rightarrow \mu(K_{j-1}) \geq N I(f_j)$.

Now $\left| I(f) - \int f \right| = \left| \sum_{j=1}^{N} I(f_j) - \int f_j \right|$

$\leq \frac{\mu(K_0) - \mu(K_N)}{N} \leq \frac{\mu(K_0)}{N} \to 0$