Basic properties of Convex functions.

① Thm 6.7 of EG: $f$ be a convex function on $\Omega$ - convex domain or $\mathbb{R}^n$

(i) $f$ is locally Lipschitz.

(ii) $\exists$ $C = C(n)$

$$\sup_{B(x, \frac{r}{2})} |f| \leq C \fint_{B(x, r)} |f| \, dy$$

(iii)
$$\operatorname*{ess\,sup}_{B(x, \frac{r}{2})} |Df| \leq \frac{C}{r} \fint_{B(x, r)} |f| \, dy$$

(iv) If $f \in C^2 \Rightarrow D^2 f \geq 0$.

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Before we start let's address some technical details

Ⓐ $f \in W^{1,1}_{loc}(\Omega)$

We have shown that $f_\varepsilon(x) = \displaystyle\int_{B(0, \varepsilon)} f(\underbrace{x-y}_{z}) \, j_\varepsilon(y) \, dy$

$$= \int_{B(x, \varepsilon)} f(z) \, j_\varepsilon(x-z) \, dz$$

$K \subset\subset \Omega$

$$f_\varepsilon(x) \longrightarrow f \text{ in } L^1(K)$$

& $\partial_j f_\varepsilon(x) \longrightarrow \partial_j f$ in $L^1(K)$

$\quad \parallel$

$(\partial_j f)_\varepsilon$

On the other hand, $\quad f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} f(z)\, j\left(\frac{x-z}{\varepsilon}\right) dz$

$$\underbrace{f_\varepsilon(x) - f(x)} = \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \left(f(z) - f(x)\right) \boxed{j\left(\frac{x-z}{\varepsilon}\right)} dz$$

$$\Rightarrow \left| f_\varepsilon(x) - f(x) \right| \leq C \fint_{B(x,\varepsilon)} |f(z) - f(x)|\, dz \longrightarrow 0$$

if $x$ is a Lebesgue point.

Similarly $\quad \left| (\partial_j f_\varepsilon)(x) - \partial_j f(x) \right| \longrightarrow 0$ as $\varepsilon \to 0$.

(B) Morrey's estimate

$$|f(y) - f(z)| \leq C r^{1-\frac{n}{p}} \left( \int_{B(x,r)} |Df|^p \right)^{1/p}$$

a.e $\quad y, z \in B(x, r)$

$$A_\varepsilon(y) = \fint_{B(y,\varepsilon)} f(w)\, dw = \frac{1}{\varepsilon^n |B(0,1)|} \int_{B(0,\varepsilon)} f(y+w)\, dw$$

$$A_\varepsilon(z) = \frac{1}{\varepsilon^n |B(0,1)|} \int_{B(0,\varepsilon)} f(z+w)\, dw$$

$$\Rightarrow \quad \underbrace{|A_\varepsilon(y) - A_\varepsilon(z)|}_{} \leq \frac{1}{\varepsilon^n |B(0,1)|} \int_{B(0,\varepsilon)} \underbrace{|f(y+w) - f(z+w)|}_{} \, dw$$

$$\leq C r^{1-\frac{n}{p}} \frac{1}{\varepsilon^n |B(0,1)|} |\varepsilon^n |B(0,1)||$$

$\forall\, y, z \in B(x, \frac{r}{2})$

$$\Rightarrow \quad \boxed{|A_\varepsilon(y) - A_\varepsilon(x)| \leq C |x-y|^{1-\frac{n}{p}}}$$

$\Rightarrow$ If we replace $f(y)$ by $\lim_{\varepsilon \to 0} A_\varepsilon(y)$

$\Rightarrow$ $f$ will be Hölder continuous.

(c) Thm: $f \; \Omega \to \mathbb{R}$

$f$ is locally Lip $\Longleftrightarrow$ $f \in W^{1,\infty}_{loc}(\Omega)$

[My last notes contain an example for global result].

($\Rightarrow$) $\forall \; V \subset\subset \Omega$

$$g_i^h := \frac{f(x+he_i) - f(x)}{h}$$

$$\sup_V |g_i^h| \leq C. \qquad (*)$$

Passing to subsequence $\quad g_i^h \longrightarrow g_i \quad$ in $L^p_{loc}$

$$\Rightarrow \int f(x) \frac{\phi(x-he_i) - \phi(x)}{h}$$

$$= \int \frac{f(x+he_i) - f(x)}{h} \phi \qquad \phi \in C_c^\infty(\Omega)$$

$$\Rightarrow \quad \int -f(x)\frac{\partial \phi}{\partial x_i} = \int g_i \phi$$

$\Rightarrow$ $g_i$ is the Distributional derivative of $f$.

$g_i \in L^\infty_{loc}$ by the estimate $(*)$.

$(\Leftarrow)$ Assume $x, y \in B(\eta)$ $\quad B(2\eta) \subset\subset \Omega$

Then

$$f^\varepsilon(x) - f^\varepsilon(y) = \int_0^1 \frac{d}{dt} f^\varepsilon(tx+(1-t)y)$$

$\varepsilon << 1$

$$= \int_0^1 \langle Df^\varepsilon|_{y+t(x-y)}, x-y\rangle$$

$$|f^\varepsilon(x) - f^\varepsilon(y)| \leq \int_0^1 \|D^\varepsilon f\|_{L^\infty(B(2\eta))} |x-y|.$$

$\Rightarrow$ $|f^\varepsilon(x) - f^\varepsilon(y)| \leq C|x-y|.$ $\quad\Rightarrow\quad$ $|f(x) - f(y)| \leq C|x-y|$ a.e. $x, y \in B(\eta)$.

Replace $f(x)$ by $\lim_{r\to 0} A_r(x)$, if needed $\Rightarrow$ $|f(x)-f(y)| \leq C|x-y|$

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(2) Proof of Thm 6.7.

We assume $0 \in \mathrm{Dom}(f)$.

Pick a small cube. $C = \square = \prod_{i=1}^n [-L, L] \subset \mathrm{Dom}(f)$

We have $\forall\, x \in C$

$$x = \sum \lambda_i v_i \qquad \sum \lambda_i = 1 \qquad \lambda_i \geq 0$$

$\Rightarrow$ $f(x) \leq \sum \lambda_i f(v_i)$ $\quad\Rightarrow\quad$ $f(x) \leq C_1 \quad \forall x \in C$

Now $\quad 0 = \frac{1}{2}x + \frac{1}{2}(-x)$

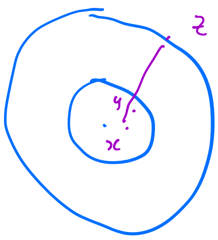
$\Rightarrow \quad f(0) \leq \frac{1}{2}f(x) + \frac{1}{2}f(-x) \leq \frac{1}{2}f(x) + \frac{1}{2}C_1$

$\Rightarrow \quad f(x) \geq -C_1' \quad \Rightarrow \quad |f|$ is locally bounded.

Now we try to show local Lipschitz property:

(i) : $\forall\ x, y \in B(0, r) \qquad B(0, 2r) \subset \mathrm{Dom}(f)$.

From $x$ to $y$ walk to $z \in \partial B(0, 2r)$.




$\Rightarrow \qquad y = \lambda z + (1-\lambda)x$

$\Rightarrow \quad y - x = \lambda(z - x)$

$u = \frac{|x-z|}{|y-x|} > 1 \qquad \Rightarrow \lambda = \frac{1}{u}$

$f(y) \leq \lambda f(z) + (1-\lambda) f(x)$

$\Rightarrow \quad f(y) - f(x) \leq \lambda\big(f(z) - f(x)\big)$

$= \frac{|x-y|}{|x-z|}\big(f(z) - f(x)\big) \leq |x-y|\big[2C_2\big]$

$|x-z| \geq 2r - |x| \geq r$

Let $\quad C_2 := \sup_{B(2r)} \frac{|f|}{r}$ $\qquad \square$

$\Rightarrow \quad f \in W^{1,\infty}_{loc}(\underbrace{\mathrm{Dom}(f)}_{}\ )$. $\quad$ ($\Omega$-convex $=$ $\mathrm{Dom}(f)$)

Hence $\quad Df \in L^{\infty}_{loc}$ —

For (ii) we need the mollifier approximation:

$j_\varepsilon$ — $j$ as before

$$f^{\varepsilon}(z) = \int f(z-w)\, j_{\varepsilon}(w)\, dw$$

$$\Rightarrow \quad f^{\varepsilon}(\lambda x + (1-\lambda) y) = \int f(\lambda x + (1-\lambda) y - w)\, j_{\varepsilon}(w)\, dw$$

$$= \int f(\lambda(x-w) + (1-\lambda)(y-w))\, j_{\varepsilon}(w)\, dw$$

$$\leq \lambda \int f(x-w)\, j_{\varepsilon}(w)\, dw + (1-\lambda) \int f(y-w)\, j_{\varepsilon}(w)\, dw$$

$$= \lambda f^{\varepsilon}(x) + (1-\lambda) f^{\varepsilon}(y).$$

Namely $f^{\varepsilon}$ – convex.

The idea is to prove the result for $f^{\varepsilon}$, then take $\varepsilon \to 0$ use $Df^{\varepsilon} \to Df$ in $L^{\infty}_{loc}$ & $f^{\varepsilon} \to f$ a. e. (also in $L^{\infty}_{loc}$).

For smooth function

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$\shortparallel$$

$$f(y + \lambda(x-y)) \leq \lambda(f(x) - f(y)) + f(y)$$

$$\Rightarrow \quad \frac{f(y + \lambda(x-y)) - f(y)}{\lambda} \leq f(x) - f(y)$$

Let $\lambda \to 0 \Rightarrow \quad Df(y) \cdot (x-y) \leq f(x) - f(y) \qquad (1)$

Namely $\quad f(x) \geq f(y) + Df(y)\cdot(x-y)$. $\quad$ 1

Applying Taylor expansion on

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f(tx + (1-t)y)$$

$$= \int_0^1 \langle \nabla f\big|_{tx+(1-t)y}, \; x-y \rangle \, dt$$

$$= \int_0^1 \langle \nabla f\big|_{(1-t)x+ty} \quad x-y \rangle \, dt$$

$$\boxed{g(t) = \langle \nabla f\big|_{(1-t)x+ty} \quad x-y \rangle}$$

$$= g(t)\,t\Big|_0^1 - \int_0^1 t \cdot \nabla^2 f_{(1-t)x+ty} \, (y-x, \; x-y)$$

$$= \langle \nabla f(y), x-y \rangle + \int_0^1 (1-s)\, \nabla^2 f_{sx+(1-s)y} \, (x-y, \; x-y) \, ds$$

$$f(x) \geq f(y) + \langle \nabla f(y), x-y \rangle$$

$$\Rightarrow \int_0^1 (1-s)\, \nabla^2 f_{\underset{\displaystyle y+s(x-y)}{\overset{\shortparallel}{sx+(1-s)y}}} \, (x-y, \; x-y)\, ds \geq 0$$

$x-y = \eta\xi$

$$\Rightarrow \int_0^1 (1-s)\, \nabla^2 f_{y+s\xi} \, (\xi \quad \xi)\, A^2 \geq 0$$

$\eta \to 0$ we get (iii).

For (ii), when $f$ is smooth

$$f(z) \geq f(y) + Df(y) \cdot (z-y)$$

$$\Rightarrow \quad f(y) \leq \fint_{B(y,\frac{r}{2})} f(z) \leq C \int_{B(x,\frac{r}{2})} |f(z)|\,dz$$

The other direction can be done similarly

$$\int f(z)\,\xi(y) \geq \int f(y)\,\xi(y) + Df(y)\cdot(z-y)\,\xi(y)$$

$$\Rightarrow \quad f(z)\left(\int_{B(z,r/2)} \xi(y)\right) \geq \int f(y)\left[\xi(y) - \mathrm{div}(\xi(y)(z-y))\right]$$

$$\geq -C\int_{B(z,\frac{r}{2})} |f(y)|$$

Here we use $\mathrm{div}(\xi(y)(z-y))$

$$= \frac{\partial \xi}{\partial y_j}\cdot(z-y) + \xi(y)(-1).$$

$\xi(y)$ can be choosen to be $= \begin{cases} 1 & \text{in } B(z,\frac{r}{4}) \\ 0 & \text{outside } B(z,\frac{r}{2}) \end{cases}$

& $|D\xi| \leq \frac{C}{r}$

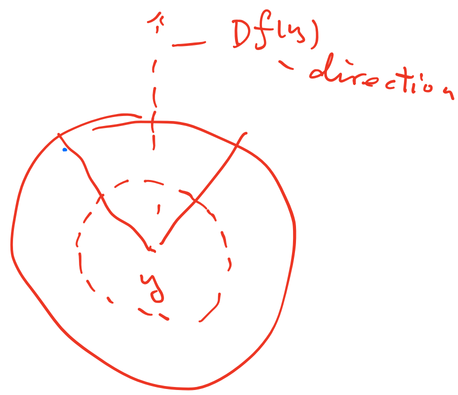
Putting together we have $\quad |f(z)| \leq C \fint_{B(x,r)} |f|(w)\,dw$

Finally for the gradient estimate.

$$f(z) \geq f(y) + \underline{Df(y)} \cdot (z-y)$$

$Df(y)$

$$S_y := \left\{ \frac{r}{4} \leq |y-z| \leq \frac{r}{2}, \quad \frac{1}{2} |Df(y)|\,|z-y| \leq Df(y)\cdot(z-y) \right\}$$




If $Df(y) \neq 0$ $\quad \cos\theta \geq \frac{1}{2}$

$\theta \leq \frac{\pi}{3}$

If $Df(y) = 0$ all $z$ satisfy

$$\Rightarrow \quad \frac{r}{8} |Df(y)| + f(y) \leq f(z)$$

$$\Rightarrow \quad \frac{r}{8} |Df(y)| \left( \int_{S_y} 1 \right) \leq \int_{B(y,\frac{r}{2})} |f(y) - f(z)|$$

$$\Rightarrow \quad |Df(y)| \leq \frac{C}{r} \fint_{B(x,r)} |f(w)|\, dw.$$