① Statement of theorems

Ⓐ Rademacher's theorem: For $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a Lipschitz function, then f is a.e differentiable.

Remark: Local nature. Only need locally sensible assumptions.

$$|f(x)-f(y)| \leq C|x-y| \qquad C = \mathrm{Lip}(f).$$

f is differentiable at x if $\exists\ L$

$$\| f(x+h) - f(x) - L(h) \| = o(|h|)$$

$$h \in \mathbb{R}^n$$

as $|h| \to 0$.

or $\quad \| f(x) - f(y) - L(x-y) \| = o(|x-y|)$

Ⓑ For $n < p < \infty$, $\exists\ C = C(p, n)$

$$|f(y) - f(z)| \leq C\, r \left(\fint_{B(x,r)} |Df|^p \right)^{\frac{1}{p}} \qquad (*)$$

$\forall$ a.e $y, z \in B(x, r)$.

Corollary: $|f(y) - f(x)| \leq C\, |y-x|^{1-\frac{n}{p}} \left(\int_{B(x,r)} |Df|^p \right)^{\frac{1}{p}}$, if $y \in \partial B(x, r)$ $\quad |y-x| = r$

$\Rightarrow$ If $B(x, r) \subset \Omega \Rightarrow$

$$\frac{|f(y) - f(x)|}{|y-x|^\alpha} \leq \left(\int_\Omega |Df|^p \right)^{\frac{1}{p}}$$

Namely we have f is Hölder Continuous.

Stronger version of R Thm: $\forall f \in W^{1,p}_{loc}(\mathbb{R}^n)$

$n < p \leq \infty$

$\Rightarrow$ f is a.e differentiable.

The proof is due to D. Adams. [a UCSD postdoc]

② Proof of Ⓐ by Ⓑ.

Let $g(y) = f(y) - f(x) - Df(x) \cdot (y-x)$

$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\langle Df(x), y-x \rangle}$.

We have $\int_K |Df(z)|^p < +\infty$ due to the assumption

Moreover by Lebesgue differentiation:

$$\lim_{r \to 0} ⨍_{B(x, r)} |Df(z) - Df(x)|^p = 0 \qquad \text{a.e } x \in \Omega$$

$$D_y g(y) = Df(y) - Df(x).$$

By (B)

$$|g(y) - g(x)| \leq C\,|x-y| \left(\fint_{B(x,r)} |Dg(z)|^p \right)^{\frac{1}{p}}$$

$$\Rightarrow \quad \frac{|f(y) - f(x) - Df(x)(y-x)|}{|y-x|} \leq C \left(\fint_{B(x,r)} |Dg(z)|^p \right)^{\frac{1}{p}} \quad r = |y-x|$$

$$= o(1) \quad \text{as } r \to 0$$

Hence (A) holds a.e $x \in \Omega$.

(3) Proof of (A):

A Lemma: $f \in C^1(B(x,r))$

$$\int_{B(x,r)} |f(y) - f(z)|^p \, dy \leq C\, r^{n+p-1} \int_{B(x,r)} |Df|^p(y)\, |y-z|^{1-n} \, dy$$

$\forall\, y, z \in B(x,r)$

Apply to $p=1$ case

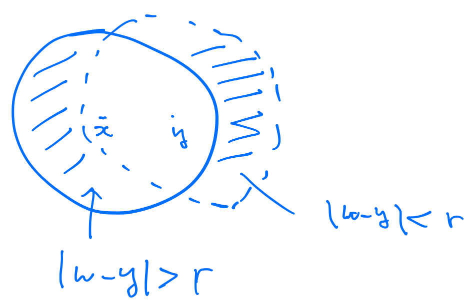
$$|f(y)-f(z)| \le \fint_{B(x,r)} |f(y)-f(z)|\,dw$$

$$\le \fint_{B(x,r)} |f(y)-f(w)| + |f(z)-f(w)|\,dw$$

$$\le C \int_{B(x,r)} |Df(w)|\left(|w-y|^{1-n} + |z-w|^{1-n}\right) dw$$

$$\le C\left(\int_{B(x,r)} |Df(w)|^p\right)^{\frac{1}{p}} \left\{ \left[\int_{B(x,r)} |w-y|^{-(n-1)\cdot\frac{p}{p-1}}\right]^{\frac{p-1}{p}} + \left[\int_{B(x,r)} |z-w|^{-(n-1)\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \right.$$

$$\int_{B(x,r)} |w-y|^{-(n-1)\frac{p}{p-1}}\,dw$$



$$= \int_{B(x,r)\cap B(y,r)} + \int_{B(x,r)\setminus B(y,r)} \le$$

$$\le \int_{B(x,r)\cap B(y,r)} |w-y|^{-(n-1)\frac{p}{p-1}}\,dw + \int_{B(y,r)\setminus B(x,r)} \text{—}$$

$$= \int_0^r s^{n-1-(n-1)\frac{p}{p-1}} \, ds < +\infty$$

Since $n-1-(n-1)\frac{p}{p-1} > -1 \iff n-(n-1)\frac{p}{p-1} > 0 \iff n(p-1)-(n-1)p > 0$
$\iff p-n > 0$

$$= C_{n,p} \, r^{n-(n-1)\frac{p}{p-1}}$$

Hence

$$|f(y)-f(z)| \leq C \left[r^{n-(n-1)\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \left(\int_{B(x,r)} |Df|^p \right)^{\frac{1}{p}}$$

$$= C \, r^{n-\frac{n}{p}-(n-1)} \left(\int_{B(x,r)} |Df|^p \right)^{\frac{1}{p}}$$

$$= C \, r^{1-\frac{n}{p}} \left(\int_{B(x,r)} |Df|^p \right)^{\frac{1}{p}}.$$

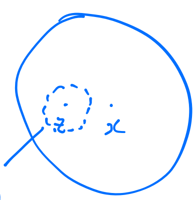
$W^{1,p}$ case follow by approximation!

④ The proof of Lemma

$$f(y)-f(z) = \int_0^1 \frac{d}{dt} f(ty+(1-t)z)$$

$$= \int_0^1 \langle \nabla f_{z+t(y-z)} \, , \, y-z \rangle \, dt$$

$$\leq |y-z| \int_0^1 |Df|_{z+t(y-z)} \, dt$$



$\partial B(z,s)$

$$\Rightarrow \quad |f(y)-f(z)|^p \leq |y-z|^p \int_0^1 |Df|^p_{z+t(y-z)}\, dt$$

$$\int_{\partial B(z,s)\cap B(x,r)} |f(y)-f(z)|^p \, dA(y) \leqslant s^p \int_0^1 \int_{\partial B(z,s)\cap B(x,r)} |Df|^p_{z+\underbrace{t(y-z)}} \, dt\, dA \quad = w$$

$y \in \partial B(z,s)$

$\Leftrightarrow w \in \partial B(z,st)$

$$= s^p \int_0^1 \frac{1}{t^{n-1}} \int_{\partial B(z,ts)\cap B(x,r)} |Df|^p(w)\, dA_w \quad dt$$

$|w-z| = ts$

$$= s^{n-1} s^p \int_0^1 \int_{\partial B(z,ts)\cap B(x,r)} |Df|^p(w) \; |w-z|^{-(n-1)} dt$$

$(\leqslant)$

$$= s^{n-1+p} \int_0^1 \int_{\partial B(z,ts)\cap B(x,r)} |Df|^2(w) \; |w-z|^{-(n-1)} \, dA(w) \; dt$$

$\tau = ts \qquad dt = \frac{d\tau}{s}$

$$= s^{n-1+p-1} \int_{B(z,s)\cap B(x,r)} |Df|^2(w) \; |w-z|^{-(n-1)} \, dw$$

Now integrating along s — $(0, 2r)$

$$\Rightarrow \int_{B(x,r)} |f(y)-f(z)|^p \leqslant \int_{B(x,r)} |Df|^2(w)\, |w-z|^{-(n-1)} dw$$

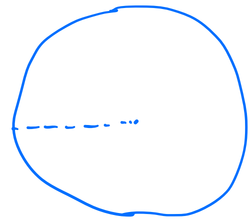
$$\cdot \int_0^{2r} s^{n-2+p}\, ds$$

$\square$

For symmetric version of Lemma see Ch10 of Krylov's book on "Lectures on Elliptic & Parabolic Eqns in Sobolev spaces"

(Ch10.)

Example.



$B(0,1) \setminus [-1, 0]$

$$f(x, y) = \sqrt{x^2+y^2}\ \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \tan^{-1}\left(\frac{y}{x}\right) + \sqrt{x^2+y^2}\ \frac{1 \left(\frac{y}{-x^2}\right)}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{\sqrt{x^2+y^2}} \left[x \tan^{-1}\left(\frac{y}{x}\right) - y \right] \qquad \in L^\infty$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \tan^{-1}\left(\frac{y}{x}\right) + \sqrt{x^2+y^2}\ \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{\sqrt{x^2+y^2}} \left[y \tan^{-1}\left(\frac{y}{x}\right) + x \right] \qquad \in L^\infty$$

Hence $f \in W^{1,\infty}(\Omega)$

But f is Not Lipschitz.