1 Bessel's inequality & Completeness of an orthonormal basis.

Bessel's inequality:

If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, then $\forall x$

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2$$

Pf: If $\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ has uncountably many > 0

Then $\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 = +\infty$.

Since if $A_n := \{\alpha \mid |\langle u_\alpha, x\rangle| > \frac{1}{n}\}$

$A_+ := \{\alpha \mid |\langle u_\alpha, x\rangle| > 0\}$ $\quad A_+ = \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow$ One of A_n is uncountable $\Rightarrow$

$$\sum_{\alpha\in A_n} |\langle x, u_\alpha\rangle|^2 \geq |A_n| \cdot \frac{1}{n}$$

On the other hand, $\forall F \subset A$ finite subset

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2$$

This is obvionse $\| x - \sum_{\alpha\in F} \langle x, u_\alpha\rangle u_\alpha \| \geq 0$

$\Rightarrow$ $\|x\|^2 - \sum_{\alpha\in F} |\langle x, u_\alpha\rangle|^2 \geq 0$

Hence $\sum_{\alpha \in A} |\langle x\ u_\alpha \rangle|^2$ is understood as

$$\sup_{\substack{F \subset A \\ F \text{ finite}}} \left\{ \sum_{\alpha \in F} |\langle x.\ u_\alpha \rangle|^2 \right\}$$

<u>Thm 5.27</u>: If $\{u_\alpha\}_{\alpha \in A}$ is an orthonormal set, the following are equivalent

(a) Completeness $\langle x\ u_\alpha \rangle = 0.\ \forall \alpha \in A \Rightarrow x = 0$

(b) $\|x\|^2 = \sum_{\alpha \in A} |\langle x\ u_\alpha \rangle|^2$

(c) $\forall x \in \mathcal{H}.\quad x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha$

RHS. only countably many $\neq 0$. & Converges. in $\|\ \|$.

<u>Pf</u>: (a) $\Rightarrow$ (c)

$\forall x, \quad \sum |\langle x, u_{\alpha_j} \rangle|^2 \leq \|x\|^2$, only countably many u_{α_j}. $\langle x\ u_{\alpha_j} \rangle \neq 0$.

$$\Rightarrow \sum_{j \geq n}^{n+m} |\langle x, u_{\alpha_j} \rangle|^2 \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \left\| \sum_{j \geq n}^{n+m} \langle x\ u_{\alpha_j} \rangle u_{\alpha_j} \right\|^2 \longrightarrow 0$$

Hence $\sum \langle x\ u_{\alpha_j} \rangle u_{\alpha_j} \longrightarrow y$

Now consider $\langle y\ u_{\alpha_k} \rangle = \langle x\ u_{\alpha_k} \rangle$

$\Rightarrow y = x$ by (a). Hence $x = \sum \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$.

(c) $\Rightarrow$ (b) $\quad x = \sum_{j=1}^{\infty} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$

means
$$\| x - \sum_{j=1}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \|^2 \longrightarrow 0 \quad \text{as } n \to \infty$$
$$\|$$
$$\|x\|^2 - \sum_{j=1}^{n} |\langle x, u_{\alpha_j} \rangle|^2 \longrightarrow 0 \Rightarrow \|x\|^2 = \sum_{j=1}^{\infty} |\langle x, u_{\alpha_j} \rangle|^2$$

(b) $\Rightarrow$ (a) It is obvious!

Corollary: Every Hilbert space has an orthonormal basis.

Pf: Defn $\{u_\alpha\}_{\alpha \in A}$ is called an orthonormal basis if the (a) - (c) holds above.

$$\mathcal{O} := \{ \{u_\alpha\}_{\alpha \in A} \text{ orthonormal set} \}.$$

A $\|$ — abbreviated.

$A \gtrsim B$ if $A \supset B$. linearly order set clearly has an upper bound. $\{A_j\}$ $\bigcup_{j \in J} A_j$ is an upper bound.

$\Rightarrow \exists$ a maximal element. $\{u_\alpha\}_{\alpha \in A_m}$

Claim: $\{u_\alpha\}_{\alpha \in A_m}$ satisfies (a).

Otherwise $\exists\, x \in \mathcal{H}$. $\langle x, u_\alpha \rangle = 0 \quad \forall\, \alpha \in A_m$
$x \neq 0$ Let $\hat{x} = \frac{x}{|x|}$ Then $\{u_\alpha\}_{\alpha \in A_m} \cup \{\hat{x}\}$ is

an orthonormal set which is $\gtrsim A_m$.

(2) Radon measures.

Here we are on X. LCH.

Eg. $X = \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, n-dim smooth mfld.

μ a measure on X, μ is assumed to be Borel.

Namely $(\mathcal{M}, \mu)$ $\forall B$ Borel set $B \in \mathcal{M}$.

Borel sets. σ-algebra generated by open sets.

μ is called a Radon measure if

(i) μ is outer regular on Borel. $\iff$ $\mu(E) = \inf_{E \subset U \ U \text{ open}} (\mu(U))$

(ii) Inner regular on open sets $\iff$ $\mu(O) = \sup_{K \subset U \ K \text{ compact}} (\mu(K))$

(iii) $\mu(K) < \infty$ $\forall K$ compact

Thm: If I is a positive linear functional on $C_c(X)$

$\Rightarrow \exists!$ a Radon measure μ

$$I(f) = \int f \, d\mu \quad \forall f \in C_c(X).$$

Moreover, (7.3) (7.4) hold.

This is the result Rudin proves at the very beginning of his R&C analysis.

We shall focus on its proof next time.