Hilbert spaces:

over $\mathbb{C}$ or $\mathbb{R}$

$\langle x, y\rangle = \overline{\langle y, x\rangle}$

$\langle a x_1 + b x_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle$

$\langle x, x\rangle \geq 0$, "=" iff $x=0$

pre-Hilbert

if $\|x\| = \sqrt{\langle x, x\rangle}$ is complete, then $\mathcal{H}$ is called a Hilbert space.

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Basics: A Schwarz inequality

B Continuity of $\|\ \|$ & $\langle\cdot,\cdot\rangle$

C parallelgram Law.

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

D Pythagorean thm: $x = \sum_{j=1}^{k} x_j$ $\{x_j\}$ $x_i \perp x_j$ $i \neq j$

$\Rightarrow \|x\|^2 = \sum \|x_j\|^2$

The proofs are all trivial.

2 Distance of a point to a closed convex set.

$K \subset \mathcal{H}$. $x \bar{\in} K$

$$\mathrm{dist}(x, K) := \inf_{y \in K} \|x-y\| := \alpha$$

By definition. $\exists\ y_n \in K$ $\|y_n - x\| \longrightarrow \alpha$ $\|y_n - x\| \geq \alpha$.

Claim: $\{y_n\}$ is Cauchy.

$$\|y_n - y_m\|^2 + \underbrace{\|y_n - x + y_m - x\|^2}_{=4\|\frac{y_n+y_m}{2} - x\|} = 2\|y_n - x\|^2 + 2\|y_m - x\|^2$$

$$\Rightarrow \quad \|y_n - y_m\|^2 \le 2\Big(\underbrace{\|y_n - x\|^2}_{\downarrow \alpha^2} + \underbrace{\|y_m - x\|^2}_{\downarrow \alpha^2}\Big) - 4\alpha^2 \longrightarrow 0.$$

Hence $\exists\, y_*$ $\quad y_n \to y_*$

$\Rightarrow$ (A) $\exists\, y_* \in K$ if K is closed, such that $\alpha = d(x, K)$
$= \|x - y_*\|.$

(B)

y_* is unique.

$$\|y_1 - x\|^2 = \alpha^2 = \|y_2 - x\|^2$$

$$\Rightarrow \quad \|y_1 - y_2\|^2 + 4\underbrace{\left\|\frac{y_1+y_2}{2} - x\right\|^2}_{\ge 4\alpha^2} = \underbrace{2\|y_1 - x\|^2 + 2\|y_2 - x\|^2}_{= 4\alpha^2}$$

$\Rightarrow \quad y_1 = y_2.$ $\square$.

Application to $\mathcal{M} \subset \mathcal{H}$ - Closed subspace

$$\mathcal{M}^\perp := \{ y \in \mathcal{H} \mid \langle y\ x \rangle = 0 \quad \forall\, x \in \mathcal{M} \}$$

$\exists$ Linear projection P & Q $\quad \forall\, x \in \mathcal{H}$ $\ x = Px + Qx.$

$Px \in \mathcal{M}.$ $\ Qx \in \mathcal{M}^\perp.$

Pf: Define $Px \in \mathcal{M}$ such that $d(x, \mathcal{M}) = \|x - Px\|.$

Px is unique.

$$\langle x - Px, z \rangle = 0 \quad \forall\, z \in \mathcal{M}.$$

If not W.L.G assume $\langle x-Px \; z\rangle$ real.

$\Rightarrow$ Consider $f(t) = \|x-Px-tz\|^2$ at $t=0$ attains its minimal
t real

$\Rightarrow 0 = f'(0) = \langle x-Px \; z\rangle = 0.$

P is linear
(Q)

$P(x+y) + Q(x+y) = x+y = Px+Qx + Py+Qy$

$\Rightarrow \underbrace{P(x+y)-Px-Py}_{\in M} = \underbrace{Qx+Qy-Q(x+y)}_{\in M^\perp}$

$\Rightarrow$ All $=0$.

③ Riesz-Representation:

$\forall f_{x_0} \in \mathcal{H}^* \;\; \exists z \qquad f(x) = \langle x, z\rangle.$

Pf: $M := f^{-1}(0)$. closed proper subspace.

$\exists u \in M^\perp \quad \|u\| = 1$

Then consider $w := f(x)u - f(u)x$

$f(w) = 0 \Rightarrow w \in M. \Rightarrow$

$\langle w, u\rangle = 0 \Rightarrow \quad f(x)\|u\|^2 - f(u)\langle x, u\rangle = 0$

$\Rightarrow f(x) = \langle x, f(u)u\rangle = \langle x, z\rangle \qquad \square$