① LCH

$\underbrace{\text{Locally}}_{/\!/}$ compact Hausdorff.

$\forall x \in X.$ $\exists N_x$ compact & $\mathring{N}_x \ni x.$ N_x is called a compact neighborhood.

Hausdorff: $\forall x, y \in X.$ $\exists \underset{x}{U_x},\ \underset{y}{V_y}$ $U_x \cap V_y = \phi.$

E.g. $\mathbb{R}^n.$ Any smooth manifold.

Non-examples: $X = \mathbb{R}$ $\mathcal{U} = \{ \mathbb{R} \setminus \{x_1, \dots x_k\} \}$

↑ cofinite topology.

NoT Hausdorff,

$(X, \|\cdot\|)$ — An infinite dimensional normed space. NoT locally compact.

Reason on LCH. $\exists$ good functions such that one can obtain good measures — Radon measures.

Two theorems. X a LCH space

Ⓐ Urysohn's Lemma: $\forall K \subset U \subset X.$ K compact U open

$\exists f \in C(X, [0,1]).$ $f|_K = 1.$ $\operatorname{Supp} f \subset \overline{V} \subset U.$ ($\overline{V}$ compact)

$\operatorname{Supp} f = \overline{\{f \neq 0\}}$

Ⓑ. Partition of the unity.

K compact. $\bigcup_{j=1}^{\ell} U_j$ a cover of $K.$ U_j open.

$\exists \{g_j\}$ $\operatorname{Supp} g_j$ is compact in $U_j.$ $g_j \in C(X, [0,1])$

& $\sum_{j=1}^{\ell} g_j = 1$ on $K.$

These two results are very useful.

② Basics on compact subsets of X. (LCH)

Lemma 1: If $F \subset X$ is compact & X is Hausdorff

$x \bar{\in} F \Rightarrow \exists\ U, V \quad x \in U. \quad F \subset V \quad U \cap V = \phi.$

Pf: $\forall y \in F \quad \exists \quad V_y \quad U_y \qquad x \in U_y \quad y \in V_y \quad U_y \cap V_y = \phi$

$\Rightarrow F \subset \bigcup_{y \in F} V_y \Rightarrow \exists \quad F \subset \bigcup_{j=1}^{\ell} V_{y_j}.$

Then let $\quad V = \bigcup_{j=1}^{\ell} V_{y_j} \qquad U = \bigcap_{j=1}^{\ell} U_{y_j}.$

$\Rightarrow U \cap V = \phi \quad \& \quad x \in U.$ $\square$.

Lemma 2. X as in L-1. $F \subset X$ is compact $\Rightarrow$ F is closed.

Pf: $\forall y \bar{\in} F \Rightarrow \exists \underset{\text{open}}{U_y} \cap F = \phi \Rightarrow \bigcup_{y \in F^c} U_y = F^c$ is open.

Trivially, any closed subset of a compact space is compact.

Lemma 3. X is LCH, $U \subset X$ is open $x \in U$ $\exists$ a compact neighborhood N of x such that $N \subset U$.

Pf: $\exists$ F compact $\quad x \in \mathring{F}$. $\quad F$ is closed by Lemma 2

Let $U_1 = U \cap \mathring{F} \Rightarrow \overline{U_1} \subset F$

Hence $\overline{U_1}$ is compact.

[If $\overline{U_1} \subset U$ we are done].

Now, we consider $\quad x \in \overline{U_1} \quad \& \quad \partial \overline{U_1} (= \partial U_1) \subset \overline{U_1}$

as two subset of $\overline{U_1}$.

Note they are closed, hence compact.

$\Rightarrow \exists\ V\ \&\ W$ open $\quad V \cap W = \phi \qquad x \in V. \ \& \ \partial \overline{U_1} \subset W.$
(By Lemma 1).

$V \subset \bar{U}_1$, $W \subset \bar{U}_1$ open (as in $\bar{U}_1$). ($\bar{V} \subset \bar{U}_1$ must be compact)

Since X is LCH, V can be chosen such that $\bar{V}$ is compact.

$\bar{V} \subset W^c$ in $\bar{U}_1$ $\Rightarrow$ $\bar{V} \subset U_1$ since $\partial \bar{U}_1 \cap \bar{V} = \phi$

Let $N = \bar{V}$. we have the result.

<u>Lemma 4</u>. X is LCH, $K \subset U \subset X$ U is open K is compact. $\Rightarrow \exists V$ such that $K \subset V \subset \bar{V} \subset U$

$\bar{V}$ is compact.

Easy consequence of Lemma 3.

<u>Pf</u> $\forall x \in K$ $\exists$ N_x $x \in \mathring{N}_x$ N_x compact $\subset U$.

$\Rightarrow \exists$ N_{x_j}. $K \subset \bigcup_{j=1}^{\ell} \mathring{N}_{x_j}$.

But $\bigcup_{j=1}^{\ell} N_{x_j}$ is still closed. & compact.

$\subset U$.

3 The proof of Thms (A) & (B).

For (A) we need the standard Urysohn Lemma:

Let X be a normal space $\exists f \in C(X, [0,1])$

$\forall A, B$ closed $A \cap B = \phi$ $f|_A = 1$ & $f|_B = 0$.

Normal $\Leftrightarrow$ $\exists U$ V open $A \subset U$. $B \subset V$ $A \cap B = \phi$.

If X is compact & Hausdorff. $\Rightarrow$ normal

Since $\forall x \in A$ $\exists x \in U_x$ $V_x \supset B$. $U_x \cap V_x = \phi$ by Lemma 1.

$A \subset \bigcup_{j=1}^{\ell} U_{x_j}$. $\Rightarrow$ $U = \bigcup_{j=1}^{\ell} U_{x_j}$ $V = \bigcap_{j=1}^{\ell} V_{x_j} \supset B$.

But X is LCH. we need a Seperate Compact set.

By Lemma 4. $\exists V$. $K \subset V \subset \bar{V} \subset U$ $\bar{V}$ is compact.

Now we apply Urysohn Lemma to $\bar{V}$. $A = K$, $B = \partial \bar{V}$

$\Rightarrow \exists$ $f \in C(\bar{V}, [0,1])$ such that $f|_K = 1$. $f|_{\partial \bar{V}} = 0$

Then define: $F(x) = \begin{cases} f(x) & x \in \bar{V} \\ 0 & x \in \bar{V}^c \end{cases}$

$$F^{-1}(E) = \begin{cases} f^{-1}(E) \quad \text{closed} & \text{if } E \not\ni 0 \\ \underbrace{V^c}_{\text{closed}} \cup \underbrace{f^{-1}(E)}_{\text{closed} \subset \bar{V}} & \text{if } 0 \in E \end{cases}$$

E closed

$\left.\begin{array}{l} F(x) = 0 \text{ on } \bar{V}^c \\ F(x) = 0 \text{ on } \partial \bar{V} \end{array}\right\} \Rightarrow F(x) = 0$ on $x \in V^c = \partial V \cup \bar{V}^c$

$\square$

Proof of (B): We apply (A) many times.

$K \subset \bigcup U_j \Rightarrow \exists \bigcup_{i=}^{\ell} N_{x_i} \supset K$
$\quad\;\; \shortparallel$
$\quad\;\; U$

By Lemma 3.

$\forall x \in K$
$\exists N_x. \subset U_j$ for some U_j

Collect them $\Rightarrow$ $K \subset \bigcup_{j=1}^{k} \tilde{N}_j$.

$\tilde{N}_j \subset U_j$ Compact.

$\Rightarrow \exists h_j$ $h_j|_{\tilde{N}_j} = 1$ & $\text{supp}(h_j) \subset \bar{V}_j \subset U_j$.

Now Consider $h = \sum_{j=1}^{k} h_j \Rightarrow$ on K $h \geq 1$

$$\underset{\text{open}}{\{h>0\}} \subset \bigcup_{j=1}^{k} \tilde{N}_j$$

$\exists \quad \underset{\in C(X,[0,1])}{f} \quad f|_K = 1 \quad \& \quad \operatorname{supp} f \subset \{h>0\}$

Then $\quad h + (1-f) > 0 \quad$ on K

Outside $\{h>0\}$ $\quad h=0$, but $f=0 \Rightarrow \quad h+(1-f)>0$

Hence $\quad h+(1-f)>0 \quad$ on X

$$\Rightarrow \quad g_i = \frac{h_i}{h+(1-f)} \quad \Rightarrow \quad \operatorname{supp} g_i = \operatorname{supp} h_i \subset U_i$$

$$\sum g_i \Big|_K = \frac{\sum_{i=1}^{k} h_i}{h+(1-f)}\Bigg|_K = \frac{h}{h+(1-f)}\Bigg|_K = 1.$$