① Approximation of Sobolev functions by smooth ones.

$\exists$ Many discussions in [EG] ch4.

$$j(x) = \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1 \\ 0 & |x|\geq 1 \end{cases}$$

$$\int_{\mathbb{R}^n} j(x) = \int_{B(0,1)} j(x) = 1$$

<u>Thm 1.</u>
$\forall f \in W^{1,p}_{loc}(\Omega)$. $\quad 1\leq p\leq\infty \quad f^\varepsilon \in C^\infty(\Omega_\varepsilon)$

Then $D_j f^\varepsilon = D_j f * j_\varepsilon$

& if $p<\infty$ $\quad f^\varepsilon \to f$ in $W^{1,p}_{loc}(\Omega)$.

<u>Remark</u>: Global approximation is possible.

<u>Thm 2.</u>
<u>Namely</u> $\forall f \in W^{1,p}(\Omega) \quad \exists \quad f^k \in W^{1,p}(\Omega)\cap C^\infty(\Omega)$

$$\|f^k - f\|_{W^{1,p}} \to 0$$

If $\partial\Omega$ is Lipschitz $\Rightarrow$ $\exists$ $f_k \in W^{1,p}(\Omega)\cap C^\infty(\bar{\Omega})$

<u>pf</u>: $\quad \Omega_\varepsilon := \{x\in\Omega \mid d(x,\Omega^c)>\varepsilon\}$

$$\bigcup \Omega_\varepsilon = \Omega$$

$$f^\varepsilon(x) = \int f(x-y)\,\eta_\varepsilon(y) = \int_{B(0,\varepsilon)} f(x-y)\,\eta_\varepsilon(y)$$

make sense $\forall x\in\Omega_\varepsilon$

$$f^{\varepsilon}(x) - f(x) = \int_{B(0,\varepsilon)} \left(f(x-y) - f(x) \right) \eta_{\varepsilon}(y)$$

$\Rightarrow \forall\, V \subset\subset \Omega$

$$\left(\int_V \left| f^{\varepsilon} - f \right|^p (x)\, dx \right)^{1/p} = \int_{B(0,\varepsilon)} \left(\int_V \left| f(x-y) - f(x) \right|^p \right)^{1/p} \eta_{\varepsilon}(y)\, dy$$

Since $\quad \| \tau_y f - f \|_{L^p(V)} \to 0 \quad$ as $\quad y \to 0$

We have the result.

Now $\quad f^{\varepsilon}(x) = \int f(\underbrace{x-y}_{z})\, \eta_{\varepsilon}(y)$

$$= \int_{B(x,\varepsilon)} f(z)\, \eta_{\varepsilon}(x-z)\, dz$$

$$= \int_{\Omega} f(z)\, \eta_{\varepsilon}(x-z)\, dz$$

$$\partial_j f^{\varepsilon}(x) = \int_{\Omega} f(z)\, \underbrace{\partial_j^x \eta_{\varepsilon}(x-z)}_{-\partial_j^z \eta_{\varepsilon}(x-z)}\, dz \qquad \text{by the DCT}$$

$$= + \int (\partial_j f)\, \eta_{\varepsilon}(x-z)\, dz$$

$$= (\partial_j f)^{\varepsilon} \qquad \text{,,}\ \partial_j (f^{\varepsilon})$$

Apply the above $\Rightarrow \quad \| (\partial_j f)^{\varepsilon} - \partial_j f \|_{L^p(V)} \to 0$, as $\varepsilon \to 0$

$\Rightarrow \forall\, V \subset\subset \Omega \qquad \| f^{\varepsilon} - f \|_{W^{1,p}(V)} \to 0$

Remark: If $spt(f) \subset \Omega$ & $f \in W^{1,p}(\Omega)$

the above shows $\exists\, f^{\varepsilon} \in C^{\infty}(\Omega)$. $\| f^{\varepsilon} - f \|_{W^{1,p}} \to 0$

(2)

Thm 2's proof

$$\tilde{\Omega}_k = \Omega_{1/k} \cap B(0,k) \qquad \bigcup \Omega_k = \Omega$$

$\Omega_k \subset\subset \Omega_{k+1}$ Hence let $V_k = \Omega_{k+1} \setminus \overline{\Omega_{k-1}}$

$V_1 = \Omega_2$

$V_2 = \Omega_3 \setminus \overline{\Omega}_1$

$$\bigcup V_k = \Omega$$

Pick ζ_k such that $spt(\zeta_k) \subset\subset V_k$ $\sum \zeta_k = 1$

$0 \leq \zeta_k \leq 1$

Let ε_k small such that: $spt[(f\zeta_k)_{\varepsilon_k}] \subset\subset V_k$

$$\| (f\zeta_k)^{\varepsilon_k} - f\zeta_k \|_{W^{1,p}} \leq \frac{\varepsilon}{2^k}$$

Then $$f^{\varepsilon} := \sum_{k=1}^{\infty} (f\zeta_k)^{\varepsilon_k}$$

is well-defined since only finite sum involved.

$$\| f^{\varepsilon} - f \|_{W^{1,p}} = \Big\| \sum_{k=1}^{\infty} \left[(f\zeta_k)^{\varepsilon_k} - f\zeta_k \right] \Big\|_{W^{1,p}}$$

$$\leq \sum_{k=1}^{\infty} \| (f\zeta_k)^{\varepsilon_k} - f\zeta_k \|_{W^{1,p}} \leq \varepsilon$$

□

Remark: $\exists$ Thm $f^{\varepsilon} \in W^{1,p}(\Omega) \cap C^{\infty}(\overline{\Omega})$ $f^{\varepsilon} \to f$ in $W^{1,p}$

if $\partial\Omega$ is Lipschitz.

(3) A chain rule.

Thm 3. (6.16 of [L]). $G: \mathbb{R}^N \to \mathbb{C}$ is C^1

& ∇G is bounded & continuous

$u = (u_1(x), \dots u_N(x))$ $\quad u_i(x) \in W^{1,p}_{loc}(\Omega)$

$\Rightarrow K(x) = (G \circ u)(x) = G(u(x)) \in W^{1,p}_{loc}(\Omega)$

Moreover (i) $\partial_j K(x) = \dfrac{\partial G}{\partial w_k}(u) \dfrac{\partial u_k}{\partial x_j}$

(ii) If $u \in W^{1,p}(\Omega)$

then $K \in W^{1,p}(\Omega)$ if either $|\Omega| < \infty$ or $G(0) = 0$.

Pf: For $V \subset\subset \Omega$.

$u^{\varepsilon} \longrightarrow u$ in $W^{1,p}(V)$ as $\varepsilon \to 0$

$\exists\ \varepsilon_s' \to 0$ subsequence $\quad u^{\varepsilon_s} \longrightarrow u$ & $\partial_j u^{\varepsilon_s} \longrightarrow \partial_j u$ a.e.

Now we have

$$K^{\varepsilon_s} = G(u^{\varepsilon_s}) \qquad K^{\varepsilon_s} - K = G(u^{\varepsilon_s}) - G(u)$$

$$\Rightarrow |K^{\varepsilon_s} - K| \leq C\, |u^{\varepsilon_s} - u| \leq C \sum_{\ell=1}^{N} |u_\ell^{\varepsilon_s} - u_\ell|$$

$$\Rightarrow \|K^{\varepsilon_s} - K\|_{L^p(V)} \longrightarrow 0$$

$$\partial_j K^{\varepsilon_s} = \frac{\partial G}{\partial w_k}(u^{\varepsilon_s})\frac{\partial (u_k)^{\varepsilon_s}}{\partial x_j}$$

$$\widetilde{\partial_j K} \doteq \frac{\partial G}{\partial w_k}(u)\frac{\partial u_k}{\partial x_j}$$

$$\Rightarrow \int (\partial_j K^{\varepsilon_s})\,\phi = -\int \frac{\partial G}{\partial w_k}(u^{\varepsilon_s})\,\frac{\partial (u_k)^{\varepsilon_s}}{\partial x_j}\,\phi$$

$$\int (\widetilde{\partial_j K})\,\phi = -\int \frac{\partial G}{\partial w_k}(u)\,\frac{\partial u_k}{\partial x_j}\,\phi$$

<u>1st</u> $\displaystyle\int \Big[\underbrace{\frac{\partial G}{\partial w_k}(u^{\varepsilon_s}) - \frac{\partial G}{\partial w_k}(u)}\Big]\frac{\partial u_k}{\partial x_j}\,\phi \longrightarrow 0$ as $s \to \infty$

By DCT.

Now $\displaystyle\int \frac{\partial G}{\partial w_k}(u)\left(\frac{\partial (u_k)^{\varepsilon_s}}{\partial x_j} - \frac{\partial u_k}{\partial x_j}\right)\phi \to 0$

as $s \to \infty$ since $\| u^{\varepsilon_s} - u \|_{W^{1,p}(V)} \longrightarrow 0$

Hence we have that

$\partial_j K^{\varepsilon_s} \longrightarrow \widetilde{\partial_j K}$ as distributions

But $\langle \partial_j K^{\varepsilon_s}, \phi \rangle = \langle K^{\varepsilon_s}, -\partial_j \phi \rangle$

$\longrightarrow \langle K, -\partial_j \phi \rangle$ since $K^{\varepsilon_s} \underset{s \to \infty}{\longrightarrow} K$ in L^p.

$\Rightarrow \quad \langle \partial_j K, \phi \rangle = \langle \widetilde{\partial_j K}, \phi \rangle$ as distributions.

$\square$

(4) Thm 6.17 of LL.

$f \in W^{1,p}(\Omega) \Rightarrow |f|(x) = |f(x)| \in W^{1,p}(\Omega)$.

$$\nabla |f| = \begin{cases} \frac{1}{|f|(x)} \left(u(x) \nabla u(x) + v(x) \nabla v(x) \right) & \text{if } f(x) \neq 0 \\ 0 & f(x) = 0 \end{cases}$$

$f(x) = u(x) + \sqrt{-1}\, v(x)$.

In particular, $\nabla |f|(x) = \begin{cases} \nabla f & f > 0 \\ -\nabla f & f < 0 \\ 0 & f = 0 \end{cases}$

for real f

& $|\nabla |f|(x)| \leq |\nabla f|$ a.e.

Pf: $|f| = \sqrt{u^2 + v^2}$

Now let $G^\varepsilon(u, v) = \sqrt{\varepsilon^2 + u^2 + v^2} - \varepsilon$

$G^\varepsilon(0, 0) = 0$.

We can apply Thm 3 above.

$$\frac{\partial G^\varepsilon}{\partial u} = \frac{u}{\sqrt{\varepsilon^2 + u^2 + v^2}}, \quad \frac{\partial G^\varepsilon}{\partial v} = \frac{v}{\sqrt{\varepsilon^2 + u^2 + v^2}}$$

$\Rightarrow \quad |\nabla G^\varepsilon| \leq 1$

Hence $\Rightarrow \quad K^\varepsilon := G^\varepsilon(u, v)$

$$\int \partial_j K^\varepsilon \phi = -\int K^\varepsilon \partial_j \phi$$

$$\parallel$$

$$\int \frac{u\,\partial_j u}{\sqrt{\varepsilon^2+u^2+v^2}}\phi + \frac{v\,\partial_j v}{\sqrt{\varepsilon^2+u^2+v^2}}\phi$$

<u>observe.</u> $K^\varepsilon := G^\varepsilon(u,v) = \sqrt{\varepsilon^2+u^2+v^2} - \varepsilon$

$$\longrightarrow |f| \quad \text{a.e}$$

& $\sqrt{\varepsilon^2+u^2+v^2} - \varepsilon \leq \sqrt{u^2+v^2}$ $\left(\Leftrightarrow \sqrt{\varepsilon^2+u^2+v^2} \leq \varepsilon + \sqrt{u^2+v^2} \right)$

(Namely $K^\varepsilon \leq |f|$.)

DCT $\Rightarrow$ $\displaystyle\int K^\varepsilon \partial_j \phi \longrightarrow \int \underset{\underset{|f|\,\partial_j\phi}{\parallel}}{K}\, \partial_j \phi$

On the other hand

$$\frac{u\,\partial_j u}{\sqrt{\varepsilon^2+u^2+v^2}} + \frac{v\,\partial_j v}{\sqrt{\varepsilon^2+u^2+v^2}} \leq |\partial_j u| + |\partial_j v|$$

$\Rightarrow$ By DCT again

$$\int \left(\frac{u\,\partial_j u + v\,\partial_j v}{\sqrt{\varepsilon^2+u^2+v^2}}\right)\phi \longrightarrow \int_{|f|\neq 0} \frac{u\,\partial_j u + v\,\partial_j v}{\sqrt{u^2+v^2}}\phi$$

(5) Fundamental solutions

$$G_y(x) = -\frac{1}{|\mathbb{S}^1|} \ln(|x-y|) \qquad n=2$$

$$G_y(x) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} |x-y|^{-(n-2)} \qquad n \geq 3$$

$$|\mathbb{S}^{n-1}| = 2\pi^{n/2} / \Gamma(\tfrac{n}{2})$$

Theorem: $-\Delta G_y = \delta_y$

Proof: $\forall \phi$ compact supported C^∞ function

$$-\int G_y \Delta \phi = \phi(y)$$

$$\int c_n \left(|x-y|^{-(n-2)}\right) \Delta\phi(x)$$

$$= \lim_{\varepsilon \to 0} c_n \int_{B(y,R) \setminus B(y,\varepsilon)} \frac{\Delta\phi(x)}{|x-y|^{n-2}} dx$$

Check $\Delta_x G = 0 \quad \forall x \neq y$

$$\int_{B(x,R) \setminus B(y,\varepsilon)} G \underbrace{\Delta\phi - \Delta G}\, \phi = \int_{\partial\Omega} G\langle \nabla\phi, \nu\rangle - \langle \nabla G, \nu\rangle \phi$$

ν - exterior normal

$= 0$ on $\partial B(x,R)$ since $\phi = 0$ in the neighborhood.

Green's formula

$$\int \operatorname{div}(X) = \int \langle X, \nu \rangle$$

$$X = G \nabla \phi \Rightarrow \operatorname{div}(X) = \langle \nabla G, \nabla \phi \rangle + G \Delta \phi$$

$$\langle X, \nu \rangle = G \langle \nabla \phi, \nu \rangle$$

If $X = (\nabla G) \phi \Rightarrow \operatorname{div}(X) = \langle \nabla G, \nabla \phi \rangle + \phi \Delta G$

$$\langle X, \nu \rangle = \langle \nabla G, \nu \rangle \phi$$

$$= + \int_{\partial B(y, \varepsilon)} \frac{C_n}{|x-y|^{n-2}} \underbrace{\langle \nabla \phi, \nu \rangle}_{\text{smooth}} + \int_{\partial B(y, \varepsilon)} \left(+ \frac{\partial G}{\partial r_y} \right) \phi$$

$\longrightarrow 0$ as $\varepsilon \to 0$ $\qquad \frac{\partial G}{\partial r_y} = - \langle \nabla G, \nu \rangle$

$$G = \frac{C_n}{r_y^{n-2}} \qquad \frac{\partial G}{\partial r_y} = -(n-2) \frac{C_n}{r_y^{+(n-1)}}$$

$$\Rightarrow \int_{\partial B(y, \varepsilon)} \frac{\partial G}{\partial r_y} \phi = -(n-2) C_n \int_{S^{n-1}} \phi(y + \varepsilon \theta) \, d\theta$$

$$\xrightarrow[\varepsilon \to 0]{} = -(n-2) C_n \phi(y) |S^{n-1}| = \phi(y)$$

$n=2$ case is similar.