1 Orders

<u>A partial ordering</u>: (i) $x \leq y$, $y \leq z \Rightarrow x \leq z$

(ii) $x \leq y$ & $y \leq x \Rightarrow x = y$

(iii) $x \leq x \quad \forall x$

<u>Linear order</u>: A partial one & if $x, y \in X$ either $x \leq y$ or $y \leq x$.

<u>well-ordered</u>: Linear ordered subset & $\exists$ a minimal element

---

<u>Directed order</u>: (i) $\alpha \lesssim \alpha \quad \forall \alpha \in A$

(ii) $\alpha \lesssim \beta$, $\beta \lesssim \gamma \Rightarrow \alpha \lesssim \gamma$

(iii) $\forall \alpha, \beta \quad \exists \gamma \quad \alpha \lesssim \gamma$, & $\beta \lesssim \gamma$.

[Partition example $x = \{x_i\}_{1 \leq i \leq n}$ $x_i \in [0,1]$

$y = \{y_j\}_{1 \leq j \leq m}$ $y_j \in [0,1]$

$x \lesssim y$ & $y \lesssim x \not\Rightarrow x = y$]

2 $\boxed{\text{Zorn's Lemma: } X \text{ is partially ordered set \& every linearly ordered subset of } X \text{ has an upper bound} \Rightarrow X \text{ has a maximal element.}}$

If X_1 compact X_2 compact

$X_1 \times X_2$ is compact one may use the definition.

or use $\langle (x^1_\alpha, x^2_\alpha) \rangle$ a net in $X_1 \times X_2$

Since $\langle x^1_\alpha \rangle$ has a subnet B, $\langle x^1_{\alpha_\beta} \rangle$ converges to x^1_∞

$\Rightarrow$ $\langle x^2_{\alpha_\beta} \rangle$ has a subnet C $\langle x^2_{\alpha_{\beta_\gamma}} \rangle$ converges to x^2_∞

[Note if $\langle x_\beta \rangle \to x$

$\gamma \to \beta_\gamma$ is a subnet $\quad \forall\, U \ni x.\ \exists\, \beta_0 .\ \forall \beta \gtrsim \beta_0\ \ x_\beta \in U$

For β_0 $\exists\, \gamma_0$ $\quad \gamma \gtrsim \gamma_0 \quad \beta_\gamma \gtrsim \beta_0 \Rightarrow x_{\beta_\gamma} \in U$, $\forall \gamma \gtrsim \gamma_0$

Hence $\langle x_{\beta_\gamma} \rangle \to x$ as well]

$$\Rightarrow \quad \langle (x^1_{\alpha_{\beta_\gamma}}\ \ x^2_{\alpha_{\beta_\gamma}}) \rangle \longrightarrow (x^1_\infty, x^2_\infty).$$

This proves $\prod_{i=1}^{\infty} X_i$ is compact if each of $\{X_i\}$ is compact

by Induction.

For $X^{\mathbb{C}}$ namely $\mathbb{C}$- many product of X, we need

Zorn's Lemma

Thm: $\prod_{\alpha \in A} X_\alpha$ is compact if each X_α is compact.

Pf: Only need to show $\forall\ \langle x_i \rangle_{i\in I}$ a net in $\prod_{\alpha \in A} X_\alpha$

$\langle x_i \rangle_{i \in I}$ has a cluster point.

To use Zorn's Lemma we define subproduct

$\forall\, B \subset A \qquad \prod_{\alpha \in B} X_\alpha. \qquad [\Pi_B(x)](\beta) = x(\beta) \qquad \forall\, \beta \in B$

$x \in \prod_{\alpha \in A} X_\alpha$ & $\Pi_B(x) \in \prod_{\alpha \in B} X_\alpha$.

Also we say p extends q if $p \in \prod_{\alpha \in C} X_\alpha$

$q \in \prod_{\alpha \in B} X_\alpha$ $\quad B \subset C.$ $\quad p(\alpha) = q(\alpha)$ if $\alpha \in B.$

Now define

$$\mathcal{P} := \bigcup_{B \subset A} \left\{ p \in \prod_{\alpha \in B} X_\alpha \ , \ p \text{ is a cluster point of } \langle \underset{\substack{\uparrow \\ \text{A net in } \prod_{\alpha \in B} X_\alpha}}{\pi_B(x_i)} \rangle \right\}$$

If $B = \{\alpha_1\}$ $\quad \pi_B(x_i) = x_i(\alpha_1) \in X_{\alpha_1}$

$\Rightarrow \langle x_i(\alpha_1) \rangle$ has a cluster point $\in X_{\alpha_1}$

$\Rightarrow \mathcal{P}$ is NOT empty.

Order, $\quad p_1 \geq p_2$ if p_1 extends p_2.

<u>Check</u>: A linearly ordered subset of $\mathcal{P}$ has an upper bound.

$\{p_\ell, \ell \in L\}$ is a linearly ordered subset.

$\forall\ p_{\ell_1}, p_{\ell_2}$ either p_{ℓ_1} extends p_{ℓ_2} or p_{ℓ_2} extends p_{ℓ_1}

Then let p^* extends $\{p_\ell, \forall \ell \in L.\}$

$p^* \in \prod_{\alpha \in B^*} X_\alpha \qquad B^* = \bigcup_{\ell \in L} B_\ell \qquad p_\ell \in B_\ell$

Namely $\pi_{B_\ell}(p^*) = p_\ell \qquad p^*$ is well-defined. since if

$\alpha \in B_\ell \cap B_{\ell'}$ $\quad$ either $\quad \underset{p_\ell \leq p_{\ell'}}{B_\ell \subset B_{\ell'}}$ $\quad$ or $\quad \underset{p_{\ell'} \leq p_\ell}{B_{\ell'} \subset B_\ell}$ $\qquad p^*(\alpha)$ is well-defined.

<u>Claim</u> p^* is a cluster point of $\langle \pi_{B^*}(x_i) \rangle \in \prod_{\alpha \in B^*} X_\alpha$

By the definition of product topology.

$\forall\ p^* \in U = \prod_{\alpha \in B^* \setminus \{\alpha_1 \dots \alpha_n\}} X_\alpha \times \prod_{j=1}^{n} U_{\alpha_j}$. $\quad U_{\alpha_j} \subset X_\alpha$ open

$U_{\alpha_j} \ni p^*(\alpha_j)$

Say $\alpha_j \in B_{\ell_j}$ $\quad B_{\ell_1} \dots B_{\ell_n}$ linearly ordered.

$p_{\ell_1} \qquad p_{\ell_n}$

order then say $\quad B_{\ell_1} \subset B_{\ell_2} \subset \dots \subset B_{\ell_n}$

$p_{\ell_1} \leq p_{\ell_2} \leq \dots \leq p_{\ell_n}$

$\pi_{B_{\ell_n}}(p^*) = p_{\ell_n}$ $\quad p_{\ell_n}$ is a cluster point of $\langle \pi_{B_{\ell_n}}(x_i) \rangle_{i \in I}$

Namely $\langle \pi_{B_{\ell_n}}(x_j) \rangle_{j \in I}$ is frequent in $\underset{\tilde{U}}{} \prod_{\alpha \in B_{\ell_n} \setminus \{\alpha_1 \dots \alpha_n\}} X_\alpha \times \prod U_{\alpha_j}$

$\left[\forall j_0\ \exists j \gtrsim j_0 \quad \in \tilde{U} \right]$

$\Rightarrow \quad \langle \pi_{B^*}(x_j) \rangle_{j \in I}$ is frequent in U.

Now apply Zorn's Lemma $\exists$ a maximum element $p \in \prod_{\alpha \in \overline{B}} X_\alpha$

p is a cluster point of $\langle \pi_{\overline{B}}(x_i) \rangle$.

But the two element case $\Longrightarrow \quad \overline{B} = A$.