① Orders

<u>A partial ordering</u>: (i) $x \lesssim y$, $y \lesssim z \Rightarrow x \lesssim z$

(ii) $x \lesssim y$ & $y \lesssim x \Rightarrow x = y$

(iii) $x \lesssim x \quad \forall x$

<u>Linear order</u>: A partial one & if $x, y \in X$ either $x \lesssim y$ or $y \lesssim x$.

<u>well-ordered</u>: Linear ordered subset & $\exists$ a minimal element

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<u>Directed order</u>: (i) $\alpha \lesssim \alpha \quad \forall \alpha \in A$

(ii) $\alpha \lesssim \beta$, $\beta \lesssim \gamma \Rightarrow \alpha \lesssim \gamma$

(iii) $\forall \alpha, \beta \quad \exists \gamma \quad \alpha \lesssim \gamma$, & $\beta \lesssim \gamma$.

[Partition example $\quad x = \{x_i\}_{1 \le i \le n} \quad x_i \in [0,1]$

$y = \{y_j\}_{1 \le j \le m} \quad y_j \in [0,1]$

$x \lesssim y$ & $y \lesssim x \not\Rightarrow x = y$]

② Zorn's Lemma: X is partially ordered set & every linearly ordered subset of X has an upper bound $\Rightarrow X$ has a maximal element.

If X_1 compact X_2 compact $X_1 \times X_2$ is compact one may use the definition.

or use $\langle (x^1_\alpha, x^2_\alpha) \rangle$ a net in $X_1 \times X_2$

Since $\langle x^1_\alpha \rangle$ has a subnet B, $\langle x^1_{\alpha_\beta} \rangle$ converges to x^1_∞

$\Rightarrow$ $\langle x^2_{\alpha_\beta} \rangle$ has a subnet C $\langle x^2_{\alpha_{\beta_\gamma}} \rangle$ converges to x^2_∞

[Note if $\langle x_\beta \rangle \to x$

$\gamma \to \beta_\gamma$ is a subnet $\forall\ U \ni x$. $\exists\ \beta_0$. $\forall\ \beta \gtrsim \beta_0$ $x_\beta \in U$

For β_0 $\exists\ \gamma_0$ $\gamma \gtrsim \gamma_0$ $\beta_\gamma \gtrsim \beta_0 \Longrightarrow x_{\beta_\gamma} \in U$, $\forall\ \gamma \gtrsim \gamma_0$

Hence $\langle x_{\beta_\gamma} \rangle \to x$ as well]

$\Rightarrow$ $\langle (x^1_{\alpha_{\beta_\gamma}}\ \ x^2_{\alpha_{\beta_\gamma}}) \rangle \longrightarrow (x^1_\infty, x^2_\infty)$.

This proves $\prod_{i=1}^{\infty} X_i$ is compact if each of $\{X_i\}$ is compact

by Induction.

For $X^{\mathbb{C}}$ namely $\mathbb{C}$- many product of X, we need

Zorn's Lemma

Thm: $\prod_{\alpha \in A} X_\alpha$ is compact if each X_α is compact.

2) For general case, use the Zorn's Lemma.

Key observation: Open sets in $\prod X_\alpha$ are $\prod U_\alpha$ — only finite many of them are not X_α

Say $\prod U_\alpha = \left(\prod_{\alpha_i} U_{\alpha_i}\right) \cdot \left(\prod_{\beta \neq \{\alpha_1 \cdots \alpha_n\}} X_\beta\right)$ — Used??

Now given $\langle x_\beta \rangle$ we want to find cluster point of $\langle x_\beta \rangle$

Concepts: (1) Subproduct

$p \in \prod_{\alpha \in A} X_\alpha$ — $p \in \prod_{\alpha \in A} X_\alpha$. $p(\alpha) \in X_\alpha$

$B \subset A$ — $\prod_{\beta \in B} X_\beta$ is called a subproduct $\to$ $p(\beta) \in X_\beta$

(2) Restriction:
$\pi_B(p)(\beta) := p(\beta) \quad \forall \beta \in B$

$\pi_{\{\alpha\}}(p) = p(\alpha)$

(3) $p \in \prod_{\alpha \in A} X_\alpha$

$q \in \prod_{\beta \in B} X_\beta$ — $B \subset A$

We say p is the extension of q if $\pi_B(p) = q$

Goal: $\langle x_i \rangle$ a net in $\prod_{\alpha \in A} X_\alpha$

try to show that $\langle x_i \rangle_{i \in I}$ has a cluster point.

1st: $B = \{\alpha\}$. $\langle \pi_{\{\alpha\}}(x_i) \rangle$

$\prod_{\beta \in B} X_\beta = X_\alpha$ ↓ has cluster point

Introduce order in the sets of cluster points of the subproduct.

$B \subset A$. $P_B \in \prod_{\beta\in B} X_\beta$ is a cluster point of $\underline{\langle \pi_B(x_i) \rangle}$

Union them all.

$$\mathcal{P} := \bigcup_{B \subset A} \left\{ P_B \in \prod_{\beta \in B} X_\beta, \ P_B \text{ is a cluster point of } \langle \pi_B(x_i) \rangle \right\}$$

$P_B \lneq P_{B'}$ if $P_{B'}$ is an extension of P_B

Namely $B \subset B'$ & $\pi_B(P_{B'}) = P_B$.

2nd: Zorn Lemma: Every linearly ordered subset of $\mathcal{P}$ has an upper bound.

$\Rightarrow \mathcal{P}$ has a maximum element.

Check linearly ordered set

$\{P_\ell, \ \ell \in L\}$ is a linearly ordered subset.
$\uparrow$ $\ell \subset A$

$P_{\ell_1} \leq P_{\ell_2}$ or $P_{\ell_2} \lneq P_{\ell_1}$

$\forall \ell_1, \ell_2$

Look at $\underline{B = \bigcup_{\ell \subset A} \ell \subset A}$

Then $\exists P_B$ such that P_B extends P_ℓ $\forall \ell \in A$

Well defined.

Defn: $\forall \beta \in B \Rightarrow \pi_\beta(P_B) = P_\ell(\beta)$
$\beta \in \ell$ for some $\ell \in A$

Claim: P_B is a cluster point of $\langle \pi_B(x_i)\rangle$ in $\prod_{\beta\in B} X_\beta$

$\forall U$ open subset of $\prod_{\beta\in B} X_\beta$

$$\Rightarrow U = \prod_{\beta\in B} U_\beta = \prod_{\beta\in \ell} U_\beta \cdot \Big(\prod_{\substack{\text{rest}\\ \alpha}} X_\alpha\Big)$$

$\Rightarrow$ $\langle \pi_\ell(x_i)\rangle$ has cluster point p_ℓ

$\Rightarrow \forall i_0 \ \exists\ i \gtrsim i_0$

$$\langle \pi_\ell(x_i)\rangle \in \prod_{\beta\in\ell} U_\beta$$

$$\Rightarrow \langle \pi_B(x_i)\rangle \in \Big(\prod_{\beta\in\ell} U_\beta\Big)\cdot\Big(\prod X_\alpha\Big) \in U$$

3rd: If $\bar{f}$ is a maximum element of $\mathcal{P}$. Easy to do. Use the argument for $X_1 \times X_0$.

$\Rightarrow \bar f \in \prod_{\beta\in B} X_\beta$

$B = A$

Thinking back on $X = \prod_{j=1}^{\infty} X_j$ case

$\exists$ cluster point $p_n \in \prod_{j=1}^{n} X_j$

More words are needed to make it complete.
First extends as this $\exists\ (p^1, p^2, \ldots$
such that $\langle \prod_n (x_i)\rangle$ has cluster point $(p^1, p^2, \ldots p^n)$

Define $(p_n^1, \ldots p_n^n)$ is a cluster point of $\langle (x_i^1, x_i^2, \ldots x_i^n)\rangle$

~~Next step $\exists$ subnet so that it converges to $(p_n^1, \ldots p_n^n)$~~

$\exists\ \langle (x_{i_j}^1, \ldots x_{i_j}^n)\rangle \to (p_1^n, \ldots p_n^n)$

~~$\langle x_i^{n+1}\rangle$ $\exists$ subnet convergent to p^{n+1}~~
~~$\langle x_{i_j}^{n+1}\rangle$ $\forall V$ $\exists\ j_0$ $j \gtrsim j_0$ $x_{i_j}^{n+1} \in V$~~

Then $\exists$ ~~subnet~~ cluster point $\langle x_{i_j}^{n+1}\rangle$, p^{n+1}

Hence $(p_n^1, \ldots p_n^n, p^{n+1})$ is a cluster point