① Alaoglu's Thm, Tychonoff's Thm & Net

A Thm: $B^* = \{ f \in \mathcal{X}^* \mid \|f\| \leq 1 \}$ then B^* is compact w.r.t weak*-topology.

T Thm: X_α, $\alpha \in A$ compact. TS. $\Rightarrow \prod_{\alpha \in A} X_\alpha$ is compact.

pf of A Th: $\forall x \in \mathcal{X}$ $\quad D_x := \{ z \in \mathbb{C} \mid |z| \leq \|x\| \}$

$\Rightarrow \Delta^* := \prod_{x \in \mathcal{X}} D_x$ is compact

$\forall f \in \prod_{x \in \mathcal{X}} D_x \qquad |f(x)| \leq \|x\|$

Hence if f is linear then $f \in B^*$.

Claim: In a topological space B^* is closed iff

$\forall \langle f_\alpha \rangle$ a net in Δ^*. $\quad f_\alpha \in B^*$

if $\langle f_\alpha \rangle \to f$ then $f \in B^*$.

[For metric space sequence is enough].

$B^* = \overline{B^*}$
if $B^* \subsetneq \overline{B^*}$
$\exists f \in \overline{B^*} \setminus B^*$
$\Rightarrow \exists \langle f_\alpha \rangle$ $f_\alpha \in B^*$
& $\langle f_\alpha \rangle \to f$

Net: A generalized concept (more general than sequence).

A net is a map from a directed set A to X.

A is a directed set: $\quad \alpha \precsim \alpha \quad \forall \alpha \in A$

$\alpha \precsim \beta,\ \beta \precsim \gamma \Rightarrow \alpha \precsim \gamma$

$\forall \alpha\, \beta \in A\ \exists \gamma \in A \quad \alpha \precsim \gamma\ \&\ \beta \precsim \gamma$

Eg. ① $\mathbb{N}$. $j \lesssim k$ iff $j \le k$

② $\mathbb{R} \setminus \{a\}$. $x \lesssim y$ $|x-a| \geq |y-a|$

③ Partitions $\{x_j\}_0^n$ of $[a, b]$ $a = x_0 < \cdots < x_n = b$

$\{x_j\}_0^n \lesssim \{y_k\}_0^m$ iff $\max |x_j - x_{j-1}| \geq \max |y_k - y_{k-1}|$

④ $\mathcal{N}$ all neighborhoods of $x \in X$.

$U \lesssim V$ iff $U \supset V$.

Prop: If X is a topological space $E \subset X$ & $x \in X$. then x is an accumulation point of E iff $\exists$ a net in $E \setminus \{x\}$ that converges to x & $x \in \bar{E}$

Pf: Accumulation point $Acc(E) := \{x \in X \mid \forall U \ni x$, Neighbor-hood $(U \setminus \{x\}) \cap E \neq \phi\}$

$(\Rightarrow)$ $\forall U$ $\exists$ $x_U \in U \setminus \{x\} \cap E$

$\langle x_U \rangle$ is a net $\longrightarrow x$

$[\forall V$ - neighborhood of x $\forall$ $U \gtrsim V \Rightarrow$ $x_U \in U \subset V]$

$\Rightarrow \langle x_U \rangle \to x$

$(\Leftarrow)$ If $\langle x_\alpha \rangle$ a net in $E \setminus \{x\}$

$\forall U \ni x \Rightarrow \exists \beta.$ $\forall \alpha \gtrsim \beta$ $x_\alpha \in U$

$\Rightarrow (U \setminus \{x\}) \cap E \neq \phi \Rightarrow x \in Acc(E)$. □

② Subnet $\langle x_\alpha \rangle_{\alpha \in A}$ $\langle y_\beta \rangle_{\beta \in B}$ $\langle y_\beta \rangle_{\beta \in B}$ is called a subset if $\exists$ map $\underset{\in B}{\beta} \to \alpha_\beta \in A$

such that $\forall \alpha_0 \in A \; \exists \beta_0 \in B$ s.t if $\beta \gtrsim \beta_0$ $\alpha_\beta \gtrsim \alpha_0$

& $y_\beta = x_{\alpha_\beta}$.

e.g. $k \to n_k$ $\langle x_{n_k} \rangle_{k \in \mathbb{N}}$ is a subset of $\langle x_n \rangle$

iff $n_k \to \infty$ if $k \to \infty$.

$\langle x_{n_k} \rangle$ is a subsequence if $\left(\begin{array}{l} n_k \neq n_\ell \text{ if } k \neq \ell \\ \& \; n_k < n_\ell \text{ if } k < \ell \end{array} \right)$

Prop 4.20: If $\langle x_\alpha \rangle_{\alpha \in A}$ is a net, then $x \in X$ is a cluster point of $\langle x_\alpha \rangle$ iff $\langle x_\alpha \rangle$ has a subnet that converges to x.

Cluster point: For sequence & For net.

$\{x_j\}$ - a sequence	$\langle x_\alpha \rangle$ a net
if $\forall U \ni x$ $\exists$ infinity many j $x_j \in U$	$\forall \alpha_0 \; \exists \; \alpha \gtrsim \alpha_0$ $x_\alpha \in U$

Pf: ($\Leftarrow$) $\exists$ B & $\beta \to \alpha_\beta$

satisfies $\forall \alpha_0 \; \exists \beta_0 \; \forall \beta \gtrsim \beta_0 \Rightarrow \alpha_\beta \gtrsim \alpha_0$

Then if $\langle y_\beta \rangle = \langle x_{\alpha_\beta} \rangle \longrightarrow x$

$\Rightarrow \forall U \ni x$. $\exists \beta_0$ $\beta \gtrsim \beta_0$ $y_\beta \in U$

$\Rightarrow \forall \alpha_0 \; \exists \beta_0$ $\beta \gtrsim \beta_0$ $\alpha_\beta \gtrsim \alpha_0$ $x_{\alpha_\beta} \in U$

($\Rightarrow$) It is a little tricky since one need to construct a subset

$\langle x_\alpha \rangle$ x is a cluster point $\Rightarrow$ $\forall U.$ & α_1 $\exists$ $\alpha_U \gtrsim \alpha_1$ $x_{\alpha_U} \in U$

Namely $\forall U$ $\exists$ α_U $x_{\alpha_U} \in U$. Consider all pair (α_U, U) $x_{\alpha_U} \in U$

$B := \{ (\alpha_U, U) \mid (\alpha_V, V) \gtrsim (\alpha_U, U)$ iff $\alpha_V \gtrsim \alpha_U$ & $V \gtrsim U \}$ $B \to A$

Then it is a subset since $\forall \alpha_0$, $\exists$ $\beta_0 = (\alpha_U, U)$ $(\alpha_U, U) \to \alpha_U$

(since $\exists \alpha_U \gtrsim \alpha_0$ $x_{\alpha_U} \in U$)

$\forall$ $\beta \gtrsim \beta_0$ $\alpha_\beta \gtrsim \alpha_0$

$\overset{\shortparallel}{(\alpha_V, V)} \gtrsim (\alpha_U, U)$ $\Leftrightarrow$ $\alpha_V \gtrsim \alpha_U$ & $V \subseteq U$ $\alpha_\beta = \alpha_V \gtrsim \alpha_U \gtrsim \alpha_0$

Claim $\langle x_{\alpha_\beta} \rangle \longrightarrow x$. $\forall U \ni x$, $\exists$ $\beta_0 = (\alpha_U, U)$ $x_{\alpha_U} \in U$

Now $\forall \beta \gtrsim \beta_0 \Rightarrow$ $(\alpha_V, V) \gtrsim \underset{\beta_0}{\underset{\shortparallel}{(\alpha_U, U)}}$

$x_{\alpha_V} \in V \subset U$ $\square$

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③ Compactness & nets

Theorem 4.29. Three are equivalent

(a) X is Compact (b) $\forall \langle x_\alpha \rangle$ net in X $\exists$ a cluster point

(c) $\forall \langle x_\alpha \rangle$ $\exists$ a subnet $\langle y_\beta \rangle_{\beta \in B} \longrightarrow x$.

Pf (b) $\Leftrightarrow$ (c) has been done by the above proposition.

Now we prove (a) & (b) are equivalent.

(a) $\Rightarrow$ (b). $\langle x_\alpha \rangle$ is a net. which has a cluster point x

Define $E_\alpha := \{ x_\beta \mid \beta \gtrsim \alpha \}$

$\forall \alpha_1, \alpha_2 \cdots \alpha_k$ $\exists$ $\alpha_0 \gtrsim \alpha_j$ $\Rightarrow$ $x_{\alpha_0} \in E_{\alpha_1} \cap \cdots \cap E_{\alpha_k}$.

$\Rightarrow$ $E_{\alpha_1} \cap \cdots \cap E_{\alpha_k} \neq \phi$.

Hence $\bigcap_{\alpha\in A} \overline{E}_\alpha \neq \phi$. Pick $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$

Then $\forall$ $U \ni x$, open neighbor, if $U \cap E_{\alpha_0} = \phi$

$\Rightarrow U^c \supset E_{\alpha_0} \Rightarrow \overline{E}_{\alpha_0} \subset U^c \Rightarrow U \cap \overline{E}_{\alpha_0} = \phi$

But $x \in U$, $x \in \overline{E}_{\alpha_0}$

$\Rightarrow U \cap E_{\alpha_0} \neq \phi \quad \forall \alpha_0$

Namely, $\forall \alpha_0 \quad \exists \beta \gtrsim \alpha_0 \quad x_\beta \in U$. $\square$

For (b) $\Rightarrow$ (a), Assume X is NOT Compact, $\exists$ $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$

Covers X. But no finite subcover

$B = \{ \underbrace{\{\alpha_1, \dots \alpha_k\}}_{\beta}, \alpha_j \in A \}$ finite subset.

$\beta \gtrsim \beta'$ if $\beta' \subset \beta$

$\Rightarrow \forall \beta \quad \exists \; x_\beta \; \overline{\in} \; \bigcup_{\alpha_j \in \beta} U_{\alpha_j}$.

$\langle x_\beta \rangle$ is a net.

We claim that it does NOT have a cluster point.

If x is a cluster point $\Rightarrow x \in U_{\alpha_0}$.

Then $\forall \quad \beta \supseteq \{\alpha_0\}$. $\Rightarrow x_\beta \; \overline{\in} \; U_{\alpha_0}$

$\Leftrightarrow \beta \gtrsim \{\alpha_0\}$

This is a contradiction!

For ($\Longrightarrow$) part of 4.20

$B = A \times \mathcal{U}$.

We need to specify the map $\beta \longrightarrow \alpha_\beta$.

$\beta = (\alpha, U)$.
$\uparrow$
Since x is a cluster point

$\exists \quad \alpha_\beta \gtrsim \alpha$

$x_{\alpha_\beta} \in U$

This is the map.

Check: $\langle y_\beta \rangle = \langle x_{\alpha_\beta} \rangle$ is a subnet:

$\forall \alpha_0$, consider $\beta_1 = (\alpha_0, U)$

$\exists \quad \alpha_{\beta_1} \gtrsim \alpha_0 \quad x_{\alpha_{\beta_1}} \in U$

Let $\beta_0 = (\alpha_{\beta_1}, U)$.

$\forall \beta \gtrsim \beta_0$

$\parallel$

$(\alpha', V) \gtrsim (\alpha_{\beta_1}, U)$

$\alpha_\beta \gtrsim \alpha'$ & $x_{\alpha_\beta} \in V \subset U$

Not needed yet!

$\alpha' \gtrsim \alpha_{\beta_1} \gtrsim \alpha_0$

$\Downarrow$

$\boxed{\alpha_\beta \gtrsim \alpha' \gtrsim \alpha_{\beta_1} \gtrsim \alpha_0}$

Moreover, $\boxed{\forall U \ni x}$. Consider $\exists$ $\boxed{\beta_0 = (\alpha_0, U)}$

$\forall \beta \gtrsim \beta_0$

$\parallel$

(α, V)

$\alpha \gtrsim \alpha_0$

$V \subset U$

$\boxed{y_\beta = x_{\alpha_\beta} \in V \subset U}$

$\boxed{\alpha_\beta \gtrsim \alpha \gtrsim \alpha_0}$

Not needed