The result of the vector-valued version is more general in two ways: (i) Assumption is weaker than FD & (B)
(ii) Covers vector valued functions.

, which is useful to study BV functions:
f is BV if
$f \in L^1_{loc}(\Omega)$, such that

$$T_f(\phi) := -\int f \sum_{i=1}^{n} \frac{\partial \phi^i}{\partial x^i}$$ satisfies $|T_f(\phi)| \le C_K |\phi|$

$\phi \in C_c^\infty(\Omega, \mathbb{R}^n)$, $\operatorname{spt}\phi \subset K$.

If $f \in W^1_{loc}(\Omega)$ $\quad T_f(\phi) = \sum \int \frac{\partial f}{\partial x_i} \phi^i$

$W^1_{loc}(\Omega) \subset BV_{loc}(\Omega)$

(A) $\Rightarrow$ $\exists\ \mu$ & σ $\quad |\sigma| = 1$. a.e

$\frac{\partial f}{\partial x_i} = \sigma_i \, d\mu$, Namely f has derivative as a measure μ

together with $\sigma = (\sigma_1 \cdots \sigma_n) \sim \left(\frac{\nabla f}{|\nabla f|} \right)$

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(1) Now we prove (a), (b) of (iv)

Pf of (a): $\quad \lambda(f) := \sup \{ |T(g)| \mid |g| \le f \}$

Clearly $\lambda(cf) = c\lambda(f)$

For f_1 & f_2. $\quad \lambda(f_1+f_2) := \sup \{ |T(g)| \mid |g| \le f_1+f_2 \}$

Construct $\quad g_i := \begin{cases} \dfrac{g f_i}{f_1+f_2} & f_1+f_2 > 0 \\ 0 & \text{else} \end{cases}$

$x_k \to x_0$, $(f_1+f_2)(x_0) = 0$ $\quad |g_i(x_k)| \le g(x_k) \to 0$
since $|g| \le f_1+f_2$.

$g = g_1 + g_2 \qquad |g_i| \leq f_i$

$\Rightarrow \quad |T(g)| \leq |T(g_1)| + |T(g_2)| \leq \lambda(f_1) + \lambda(f_2)$.

On the other hand, $\forall \varepsilon > 0$, $\exists \; g_i \quad |T(g_i)| + \frac{\varepsilon}{2} \geq \lambda(f_i)$

$|g_i| \leq f_i$

$\Rightarrow \qquad g = g_1 + g_2$ satisfies

$T(g) = T(g_1) + T(g_2)$

$|g| \leq |g_1| + |g_2| \leq f_1 + f_2 \quad \Rightarrow$

$|T(g)| = |T(g_1)| + |T(g_2)| \; (= T(g_1) + T(g_2))$

by flipping $g_i \to -g_i$ if needed

$\geq \lambda(f_1) + \lambda(f_2) - \varepsilon$

Hence $\quad \lambda(f) \geq \lambda(f_1) + \lambda(f_2)$.

For (b), the proof is similar to that in FD ((iv) on page 214).

But if we use FD Thm 7.2, it is easy to show

$\lambda(f) = \int f \, d\tilde{\mu}$ for $\tilde{\mu}$ – Radon measure

defined by $\quad \tilde{\mu}(U) := \sup \{ \lambda(f) \mid 0 \leq f \leq 1, \; \operatorname{spt} f \subset U \}$

with f satisfying

Claim: $\tilde{\mu}(U) = \mu(U)$

$\forall f$, in $\tilde{\mu}$-definition, consider $g \quad |g| \leq f$

$\Rightarrow$ such $g \in C_c(\Omega, \mathbb{R}^m) \quad \operatorname{spt} g \subset U \;\&\; 0 \leq |g| \leq 1$

Hence $\quad |T(g)| \leq \mu(U)$ by μ-definition

$\Rightarrow \quad \lambda(f) \leq \mu(U)$ if f is in $\tilde{\mu}$-definition

$\Rightarrow \quad \tilde{\mu}(U) \leq \mu(U)$

On the other hand, $\forall$ g in μ- definition

Let $f = |g|$ $\Rightarrow$ f satisfies $\tilde{\mu}$- definition

$$T(g) \leq \lambda(f) \leq \tilde{\mu}(U) \Longrightarrow \mu(U) \leq \tilde{\mu}(U).$$

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2 Construction of σ.

$\forall e \in \mathbb{R}^m$ $|e|=1$. $T_e(f) = T(fe)$. $\forall f \in C_c(\Omega, \mathbb{R})$.

$$\left| T_e(f) \right| \leq \left| T(\underbrace{fe}) \right| \leq \lambda(|f|) = \int |f| \, d\mu$$

$$\sup \overset{\|}{\{} |T(g)| \;\big|\; |g| \leq |f| \}$$

$g_0 = fe$ will be one

Hence $T_e(f)$ is a bounded linear function of $L^1(d\mu)$

By the duality theorem: $T_e(f) = \int f \cdot \sigma_e \, d\mu$
(Thm 6.15 of FD).

for some $\sigma_e \in L^\infty(\Omega, d\mu)$.

Applying to $e_1, \dots e_m$ standard basis of $\mathbb{R}^m$ $\Rightarrow$

$\exists$ $\sigma_1, \dots \sigma_m$

& $T_{e_i}(f) = T(fe_i)$

Hence for $g = \sum_{i=1}^m g^i e_i$ $\Rightarrow$

$$T(g) = \sum_{i=1}^m T_{e_i}(g^i) = \int \sum_{i=1}^m g^i \sigma_i \, d\mu$$

3

Now we show $|\sigma| = 1$. a.e for μ.

It suffices to show $\forall$ U - open

$$\int_U |\sigma|\, d\mu = \mu(U) \qquad (\star) .$$

We use Lusin's theorem find [Thm 7.10 of FD]

$f_k^i \nearrow \frac{\sigma_i}{|\sigma|}$ on $|\sigma| \neq 0$ as $k \to \infty$ such that $\|f_k^i\|_u \leq \left\|\frac{|\sigma_i|}{|\sigma|}\right\|_{L^\infty}$

& $f_k^i \in C_c(\Omega, \mathbb{R})$

$\Rightarrow$ Fm $f_k = (f_k^1, \dots f_k^m)$ $\quad |f_k| \leq 1$

$\operatorname{spt}(f_k) \subset U$

$$\Rightarrow \quad T(f_k) = T(f_k^i e_i) = \int (f_k^i \sigma_i)\, d\mu$$

$$\Rightarrow \quad \int_U |\sigma|\, d\mu = \lim_{k\to\infty} \int_U (f_k^i \sigma_i) = T(f_k) \leq \mu(U).$$

On the other hand

$\forall f \in C_c(\Omega, \mathbb{R}^m)$ satisfies $|f| \leq 1$. $\operatorname{spt}(f) \in U$

$$\Rightarrow \quad T(f) = \int \langle f, \sigma \rangle\, d\mu \leq \int_U |\sigma|\, d\mu$$

Hence $\mu(U) \leq \int_U |\sigma|\, d\mu$

Putting together $\Rightarrow$ $\int_U |\sigma|\, d\mu = \mu(U)$

$(\star) \Rightarrow |\sigma| = 1$ a.e

If NOT. Say $\exists\, n$ $\quad E = \underbrace{\{|\sigma| \geq 1 + \tfrac{1}{n}\}}_{f}$ $\quad \underbrace{\mu(E)}_{A} > 0$

By regularity $\Rightarrow$ $\exists\, U$ $\quad \mu(U \setminus E) < \varepsilon$

U open. Define $g = \begin{cases} f & \text{in } E \\ 0 & \text{else} \end{cases}$

$$\Rightarrow \quad \|g\|_{L^\infty} \leq \|f\|_{L^\infty} = C.$$

Since Ω is σ-compact

$\Rightarrow$ μ is σ-finite

We may replace E by $E \cap F_j$ with $\mu(F_j) < +\infty$

Hence $\mu(E \cap F_j) < +\infty$

& make the arguement on $E \cap F_j$.

$$\int_U (f-g) = \int_{U\setminus E} f \geq 0$$

$$\|$$

$$\mu(U) - \int_U g = \mu(U) - \underbrace{\int_E f} \leq \underbrace{\mu(U)}-\mu(E) - \tfrac{1}{n}\mu(E)$$

$$\leq \varepsilon - \tfrac{1}{n}\mu(E)$$

$$\int_E f \geq (1+\tfrac{1}{n})\,\mu(E)$$

$$\boxed{0 \leq \varepsilon - \tfrac{1}{n}\mu(E)}$$

Not possible if $\varepsilon \ll 1$.

has positive measure

Similarly if $E = \{ |\sigma_f| \leq 1-\frac{1}{n} \}$.

$$\int_U (f-g) = \int_{U\setminus E} f \leq C\,\mu(U\setminus E)$$

$$\|$$

$$\mu(U) - \int_E f \geq \mu(U) - \mu(E)(1-\tfrac{1}{n})$$

$$\Longrightarrow \quad \tfrac{1}{n}\mu(E) \leq (1+C)\,\underbrace{\mu(U\setminus E)}_{\leq \varepsilon}. \quad \text{for } \varepsilon \ll 1$$

This is NOT possible.