1 Topological Vector space

$\mathcal{X}$ is called a TVS

if 1) $\mathcal{X}$ is a linear space

2) $\mathcal{X}$ is a topological space

3) $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ $\quad (x, y) \to x+y$

continuous

4) $\mathbb{R} \times \mathcal{X} \to \mathcal{X}$ $\quad (\lambda, x) \to \lambda x$ is continuous.

$(\mathbb{C} \times \mathcal{X})$

<u>Prop 1</u>: If U is a neighborhood of 0 $\Rightarrow$ $U + x_0$ is a neighborhood of x_0

<u>Pf</u>: $\exists\ V$ open $\subseteq U$ $\quad 0 \in V$.

$V + x_0 \subset U + x_0$

$f: x \to x - x_0$ is a continuous map; (f^{-1} as well)

$\mathcal{X} \to \mathcal{X}$

$V + x_0 = f^{-1}(V) \Rightarrow V + x_0$ is open. $\square$

<u>E.g.</u> (i) If $\mathcal{X}$ is a normed VS $\Rightarrow$ $\mathcal{X}$ is TVS.

$\|x+y\|$ is continuous since if $x_n \to x_0$

$\|x_n + y - (x_0+y)\| = \|x_n - x_0\| \to 0$

Similarly $\|\alpha x_n - \alpha x_0\| = |\alpha|\ \|x_n - x_0\| \to 0$.

(ii) $\mathbb{R}^\infty = (x_1, \cdots, x_n, \cdots)$ $\quad x_i \in \mathbb{R}$

VS over $\mathbb{R}$

$\mathbb{R}^\infty = \mathbb{R}^{\mathbb{N}}$ – product topology.

$U_{k_1 \cdots k_r, \varepsilon} := \{ x \in \mathbb{R}^\infty \mid |x_{k_i}| < \varepsilon \quad 1 \le i \le r \}$

$\{U_{k_1 \cdots k_r, \varepsilon}\}$ generate a neighborhood base near $(0, \cdots 0)$

Then we have a topology by $\mathcal{T}$ - generated by translations of $\{U_{k_1 \cdots k_r \varepsilon}\}$.

<u>Def</u>: A set M is called bounded if $\forall\, U$ neighborhood of 0 $\exists \alpha \quad M \subset \alpha U$.

X is called locally bounded if $\exists$ a bounded open set

e.g:

$\mathbb{R}^\infty$ is NOT locally bounded.

$U_{i_1 \cdots i_r} = \{x \mid |x_1| < \varepsilon, \cdots |x_{i_r}| < \varepsilon\}$ is Not bounded

Since $U_{i_1 \cdots i_r} \not\subset \underbrace{\alpha U_j}_{\forall \alpha}$ if $j \notin i_1 \cdots i_r$

$\Rightarrow \mathbb{R}^\infty$ topology as above can NOT be given by a topology of a norm.

Any normed vector space is locally bounded.

This shows that why we need to study TVS.

2 The convergence in Weak topology, weak*- topology & Product topology.

<u>Thm</u>: (i) $f: X \to \mathbb{C} \Leftrightarrow f \in \mathbb{C}^X$

$\{f_n \in \mathbb{C}^X\}, \{f_n\} \to f$ iff $\{f_n(x)\} \to \{f(x)\}$

(ii) $\mathcal{X}$ — normed vector space $\mathcal{X}^*$ its dual

$\{f_n\} \in \mathcal{X}^*$ can be viewed as $\underbrace{\mathbb{C}^{\mathcal{X}}}_{\text{subsets}}$

$\{f_n\} \to f$ w.r.t weak*-topology iff

$\{f_n(x)\} \to \{f(x)\}$ $\forall x \in \mathcal{X}$.

(iii) As in $\mathcal{X}$. $\{x_n\} \to x$ w.r.t weak topology

iff $\forall f \in \mathcal{X}^*$ $\{f(x_n)\} \to f(x)$.

<u>pf</u>: For (i). ($\Rightarrow$) $\{f_n\} \to f \Rightarrow \forall \varepsilon > 0$

let $U_{x\varepsilon} = \{ g \mid |g-f|(x) < \varepsilon \}$ be the neighborhood of f

$\Rightarrow \exists N.$ $n \geq N$ $f_n \in U_{x\varepsilon}$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon \Rightarrow f_n(x) \to f(x)$.

($\Leftarrow$) Let $U_{x_1 \cdots x_p, \varepsilon} = \{ g \mid |g-f|(x_1) < \varepsilon, \cdots |g-f|(x_p) < \varepsilon \}$.

be a neighborhood of f (Any other one is union of such)

Since $f_n(x_j) \to f(x_j)$ $\exists N$ $n \geq N$

$|f_n(x_j) - f(x_j)| < \varepsilon$ $j = 1, \cdots r$

$\Rightarrow f_n \in U_{x_1 \cdots x_r, \varepsilon}$ if $n \geq N \Rightarrow \{f_n\} \to f$

(ii) Proof is exactly same since the weak*-topology is defined by $U_{i_1 \cdots i_r, \varepsilon} = \{ g \in \mathcal{X}^* \mid |g-f|(x_1) < \varepsilon, \cdots |g-f|(x_p) < \varepsilon \}$.

(iii) It is the dual version.

Namely ($\Rightarrow$), if $x_n \to x$ w.r.t the weak topology.

then $\forall f \in X^*$

$U_{f\varepsilon} := \{ y \mid |f(y) - f(x)| < \varepsilon \}$ is a neighborhood of x

$\Rightarrow \exists N \quad n \geq N \quad x_n \in U$

$\Rightarrow \quad |f(x_n) - f(x)| < \varepsilon \Rightarrow f(x_n) \to f(x)$.

($\Leftarrow$) If $\{f(x_n)\} \to f(x) \quad \forall f \in X^*$

for $U_{f_1 \cdots f_r \varepsilon} := \{ y \mid |f_1(y) - f_1(x)| < \varepsilon \ \cdots \ |f_r(y) - f_r(x)| < \varepsilon \}$

(is a generic neighborhood of x)

by $\{f_j(x_n)\} \to \{f_j(x)\}_{1 \leq j \leq r} \Rightarrow \exists N. \quad n \geq N$

$\Rightarrow \quad |f_j(x_n) - f_j(x)| < \varepsilon \quad 1 \leq j \leq r$

$\Rightarrow \quad x_n \in U_{f_1 \cdots f_r, \varepsilon}$ $\square$

③ A special case of Alaoglu's thm

Thm: X is seperable normed VS. $\{f_n\} \subset X^*$, continuous linear functions. If $\{\|f_n\|\}$ is bounded $\Rightarrow \exists \{f_{n_k}\}$ converges in weak*-topology.

<u>Alaoglu's Thm:</u> $B^* = \{ f \in X^* \mid \|f\| < 1 \} \subset X^*$ is compact in the weak*-topology.

<u>Two distinctions</u>: (i) In Alaoglu's thm no assumption on X

(ii) Compactness in weak*-topology $\not\Rightarrow$ subsequential convergence.

Pf: Let $\{x_n\}$ be dense subset of $\mathcal{X}$

$\Rightarrow$ $\{f_n(x_1)\}$ is bounded in $\mathbb{C}$

$\Rightarrow$ $\exists$ $\{f_{n_k}(x_1)\}$ convergent

$$f_{n_1}(x_1) \quad f_{n_2}(x_1) \quad \cdots \quad \longrightarrow$$

$\{f_{n_k}(x_2)\}$ $\exists$ subsequence bounded in $\mathbb{C}$

$$f_{n_{k_1}}(x_2), \quad f_{n_{k_2}}(x_2) \quad \cdots \quad \longrightarrow$$

Better terminology is

$$\begin{pmatrix} f_1^{(1)}(x_1) & f_2^{(1)}(x_1) & \cdots & \longrightarrow \\ f_1^{(2)}(x_2) & f_2^{(2)}(x_2) & \cdots & \longrightarrow \end{pmatrix}$$

$\{f_n^{(2)}\}$ is a subsequence of $\{f_k^{(1)}\}$

Now consider $\{f_n^{(n)}\}$ $\Rightarrow$ $\{f_n^{(n)}(x_j)\}$ is convergent

$\Rightarrow$ Define $f_\infty(x_j) = \lim\limits_{n\to\infty} f_n^{(n)}(x_j)$

& $\forall$ $x \in X$ we define

$$f_\infty(x) = \lim_{k\to\infty} f_\infty(x_{j_k}) \quad \text{if} \quad x_{j_k} \to x$$

Then if $\{x_{j_k}\}$ & $\{x'_{j_k}\}$ $\longrightarrow x$

$\Rightarrow$ $\| x_{j_k} - x'_{j_k} \| \longrightarrow 0$ as $k \to \infty$

$$\Rightarrow \quad \| f_n^{(n)}(x_{j_n}) - f_n^{(n)}(x_{j'_n}) \| \leq M \| x_{j_n} - x_{j'_n} \| \to 0$$

$\Rightarrow$ $f_\infty(x)$ is well-defined.

& $f_\infty(x+y) = f_\infty(x) + f_\infty(y)$ | $f_\infty(\alpha x) = \alpha f(x)$

$$\| f_\infty(x) \| \leq M \lim_{n\to\infty} \| x_{j_n} \| = M \| x \|$$

$\Rightarrow f_\infty \in X^*$. Moreover

$$f_\infty(x) = \lim_{n\to\infty} f_n^{(n)}(x)$$

Since $f_\infty(x) = \lim_{n\to\infty} f_n^{(n)}(x_{j_n})$ & $| f_n^{(n)}(x - x_{j_n}) |$

$$\leq M \| x - x_{j_n} \| \to 0$$

$\square$.

If weak*-topology is given by a metric then this is special case of Alaoglu's.