1. Thm 7.2 of FD: T is a positive linear functional of $C_c(X)$. X a locally compact Hausdorff space.

Then $\exists\, \mu$ - a Radon measure on X such that

$T(f) = \int f\, d\mu$.

Moreover $\forall U$ open $\mu(U) = \sup\{T(f),\ f \in C_c(X),\ f \prec U\}$

$\mu(K) = \inf\{T(f),\ f \in C_c(X),\ f \geq \chi_K\}$ $\forall K$ compact

"Positive" $T(f) \geq 0$. $\forall f \geq 0$.

Radon measure,
1) Borel
2) outer regular on Borel sets $\mu(E) = \inf \mu(U)$ $E \subset U$
3) inner regular on open sets $\mu(U) = \sup \mu(K)$, $\forall K \subset U$, K compact.
4) $\mu(K) < \infty$, $\forall K$ compact

$f \prec U$ means "$0 \leq f \leq 1$, $\operatorname{supp} f \subset U$."

Two generalizations: (A) Vector-valued functions.

Thm 1.38 of (EG - 2015 Ed). $T: C_c(\Omega, \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying: $\forall K \subset\subset \mathbb{R}^n$

$C_K := \sup\{T(f) \mid f \in C_c(\Omega, \mathbb{R}^m),\ |f| \leq 1,\ \operatorname{spt}(f) \subseteq K\} < \infty$ $(*)$

Then $\exists\, \mu$ - Radon measure & $\sigma: \Omega \to \mathbb{R}^m$ $\Omega \subset \mathbb{R}^n$

$|\sigma(x)| = 1$ a.e. μ

$T(f) = \int_{\mathbb{R}^n} \langle f, \sigma \rangle\, d\mu$ $\forall f \in C_c^\infty(\Omega, \mathbb{R}^m)$.

$\mathbb{R}^n$ can be replaced by $\Omega \subset \mathbb{R}^n$.

(B) - Generalization to distributions

Thm 6.22 of LL (One can find this & its proof in Hörmander ALPDO Vol I).

$T \in \mathcal{D}'(\Omega)$ be a positive distribution, namely

$T(\phi) \geq 0$ if $\phi \in D(\Omega)$ satisfies $\phi \geq 0$.

Then T is given by a Radon measure μ, namely

$$T(\phi) = \int_\Omega \phi \, d\mu$$

② Proof of Ⓑ

$$\left\{ \begin{array}{l} \text{Given } K \subset\subset \Omega, \ \exists \psi \text{ such that} \\ \psi = 1 \text{ on } K, \quad 0 \leq \psi \leq 1, \quad \operatorname{supp}\psi \subset \Omega \end{array} \right.$$

Uryshon Lemma on $X = \Omega$ which is LCH

By convolution (Say 1st enlarge K to $K' \subset\subset \Omega$)

ψ can be chosen to be $C_c^\infty(\Omega)$.

$$\psi = \int_{B(0,\varepsilon)} \psi'(x-y) \, j_\varepsilon(y) \, dy$$

$\varepsilon \ll 1 \implies \psi|_K = 1$ since $\psi'(x-y) = 1$ if $x \in K$, $|y| \leq \varepsilon$

if $d(x, \operatorname{supp}\psi') > \varepsilon \implies \psi(x) = 0$

Consider ϕ $\operatorname{supp}\phi \subset K$, $\quad |\phi|_u \psi - \phi \geq 0$

$$\implies T(|\phi|_u \psi - \phi) \geq 0 \implies T(\phi) \leq T(\psi) \, |\phi|_u$$

Similarly $|\phi|_u \psi + \phi \geq 0 \implies |T(\phi)| \leq T(\psi) \, |\phi|_u$

$\qquad |\phi|_u = |\phi|_{L^\infty}$

Namely $\boxed{\forall \phi \in C_c^\infty(\Omega) \quad \operatorname{supp}\phi \subset K \quad |T(\phi)| \leq C \, |\phi|_u}$ $\quad (*)$

Since $\boxed{\forall \phi_1 \text{ can be approximated by } \phi \in C_c^\infty(\Omega) \atop \in C_c(\Omega)\ \operatorname{supp}\phi_1 \subset K_1}$ $\operatorname{supp}\phi \subset K$ ($K_1 \subset \dot{K}$)

we have $(*)$ holds for all $\phi \in C_c(\Omega)$

Now T can be uniquely extended to a linear functional on $C_c(\Omega)$; pick $\phi_n \in C_c^\infty(\Omega)$ $\operatorname{supp}\phi_n \subset K$

$\phi_n \longrightarrow \phi$ in L^∞ $\{T(\phi_n)\}$ is Cauchy.

$\Rightarrow$ $T(\phi) := \lim T(\phi_n)$

Apply Thm (A) $\Longrightarrow$ (B).

One may also use FD Thm 7.2 since by the above T a positive distribution defines a $T(\phi)$ $\forall \phi \in C_c(\Omega)$.

by $\lim\limits_{n\to\infty} T(\phi_n)$ ϕ_n can be chosen to be ≥ 0

$\Longrightarrow$ $T(\phi) \geq 0$. Namely we have a positive

$T: C_c(\Omega) \to \mathbb{R}$.

3 Proof of (A):

Similar to FD Thm 7.2.

$$\mu(U) := \sup\left\{ T(g) \;\middle|\; \in C_c^\infty(\Omega, \mathbb{R}^m),\ |g| \leq 1,\ \operatorname{spt}(g) \subset U \right\}$$

$$\mu^*(E) = \inf\left\{ \mu(U) \;\middle|\; U \supset E,\ U \text{ is open} \right\}.$$

Step (i) & (ii) in FD Thm 7.2 proof can be checked.

(i) μ^* is an outer measure.

In fact $\mu(U) \leq \sum_{i=1}^{\infty} \mu(U_i)$ if $U = \bigcup_{i=1}^{\infty} U_i$

Since $\forall f \in C_c^{\infty}(\Omega, \mathbb{R}^n)$. $spt(f) \subset U$, $|f| \leq 1$

Let $K = spt(f)$. Pick finite $\{U_i\}$ say $U_1 \cdots U_k$

$K \subset \bigcup_{i=1}^{k} U_i$, Pick φ_i $\sum \varphi_i = 1$ on K

$0 \leq \varphi_i \leq 1$ $spt\,\varphi_i \subset U_i$.

$$\Rightarrow \quad f = \sum_{i=1}^{k} f\varphi_i$$

$$|T(f)| \leq \sum_{i=1}^{k} |T(f\varphi_i)| \leq \sum_{i=1}^{k} \mu(U_i)$$

$spt(f\varphi_i) \subset U_i$ $|f\varphi_i| \leq 1$

(ii) Each open set is measurable ($\Longrightarrow$ All Borel sets are measurable).

The same arguement on Page 214 of FD works.

$\forall$ E open it is easy to show

$$\mu^*(E \cap U) + \mu^*(E \setminus U) = \mu^*(E).$$

$(\leq \mu^*(E))$

$\mu(E) = \sup\{T(f) \mid \cdots \}$

Since $E \cap U$ is open $\exists$ f $spt(f) \subset E \cap U$

Such that $\mu^*(E \cap U) \leq T(f) + \varepsilon/2$

& $E \setminus spt(f)$ is also open

$\exists\, g$ $spt(g) \subset E \setminus spt(f)$ $|g| \leq 1$ $\frac{\varepsilon}{2} + T(g) \geq \mu(E \setminus spt(f))$.

$$\Rightarrow T(f+g) \geq \mu^*(E \cap U) + \mu^*(E \setminus spt(f)) - \varepsilon$$

But $|T(f+g)| \le \mu(E)$ given $|f+g| = \begin{cases} |f| & \text{on spt} f \\ |g| & \text{on spt} g \end{cases}$

$\mathrm{spt}(f+g) \subset E$.

The result on general E can be done by outer regularity.

$\boxed{\text{(iii) of FD Thm 7.2 (7.4) used positivity of } T.}$

& inner regularity of open sets.

Here we observe that $\forall K$, define $K_\varepsilon = \{x \in \Omega \mid d(x, K) < \varepsilon\}$ or $\mathbb{R}^n$

$\overline{K_\varepsilon}$ is compact. $\mu(K_\varepsilon) := \sup\{ T(f) \mid f \in C_c^\infty(\Omega, \mathbb{R}^n),\ |f| \le 1 \ \&\ \mathrm{spt}(f) \subset K_\varepsilon \subset \overline{K_\varepsilon} \}$

$\Rightarrow \mu(K_\varepsilon)$ is finite by $(*)$

$\Rightarrow \mu(K) \le \mu(K_\varepsilon) < +\infty$,

Now Thm 7.8 ($\exists$ a proof without using Thm 7.2) $\Longrightarrow$ μ is Radon (we use Thm 7.2).

Note: We do not have (7.4) since now T applies to vector-valued functions. $f \ge \chi_K$ does not make sense now.

Namely $\boxed{\text{(iii') is } \mu \text{ is Radon.}}$

(iv): $\lambda(f) := \sup\{ |T(g)| \mid g \in C_c^\infty(\Omega, \mathbb{R}^n), |g| \le f \}$

$\forall f \in C_c(\Omega),\ f \ge 0$

λ satisfies

(a) $\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2)$

(b) $\lambda(f) = \int_\Omega f \, d\mu$