Three results from LL.

1 Theorem 6.9.

Let $\Omega \subset \mathbb{R}^n$, $T \in \mathcal{D}'(\Omega)$, $\phi \in \mathcal{D}(\Omega)$. Assume $\phi_{ty} \in \mathcal{D}(\Omega)$ $0 \le t \le 1$ for some $y \in \mathbb{R}^n$.

Then $$T(\phi_y) - T(\phi) = \int_0^1 \sum y_j (\partial_j T)(\phi_{ty})\, dt \qquad (1.1)$$

$\forall\, f \in W^{1,1}_{loc}(\mathbb{R}^n)$ $\forall\, y \in \mathbb{R}^n$

$$f(x+y) - f(x) = \int_0^1 \langle y, \nabla f(x+ty) \rangle\, dt \qquad (1.2)$$

Proof: $$F(y) - F(0) = \int_0^1 \frac{d}{dt} F(ty)\, dt$$

$$= \int_0^1 \sum y_j \frac{\partial F}{\partial y_j}\Big|_{ty} dt.$$

$F(y) = T(\phi_y)$ We also have $\frac{\partial F}{\partial y_j} = \frac{\partial}{\partial y_j} T(\phi_y)$

$= (\partial_j T)(\phi_y)$. $\Longrightarrow$ (1.1)

(1.2) is a simple consequence of (1.1).

2 Thm 6.10.

$\Omega \subset \mathbb{R}^n$, $T \in \mathcal{D}'(\Omega)$, $G_i = \partial_i T \in \mathcal{D}'(\Omega)$, $\forall\, i = 1, \dots, n$.

Then the following two are equivalent

(i) T is a function $f \in C^1(\Omega)$ $\left(W^{1,1}_{loc}(\Omega)\right)$

(ii) G_i is a function $g_i \in C^0(\Omega)$ $\left(L^1_{loc}(\Omega)\right)$ $\forall\, i = 1, \dots n$.

In each case, $g_i = \frac{\partial f}{\partial x_i}$ (the classical or generalized derivative) pf: (i) $\Rightarrow$ (ii) is obvious!

(ii) $\Rightarrow$ (i): The key is to find function f which gives $T_f = T$.

$\forall\ \Omega_\delta := \{ x \in \Omega \mid \operatorname{dist}(x, \Omega^c) > \delta \}$ $\quad \delta > 0$ small.

We will consider $\phi \in \mathcal{D}(\Omega_\delta)$. $\quad \phi_y$ is defined if $|y| < \delta/2$.

$$T(\phi_y) - T(\phi) = \int_0^1 \left[\sum y_j \, (\partial_j T)(\phi_{ty}) \right] dt$$

$$= \int_0^1 \left[\sum y_j \int_{\Omega} g_j(x)\, \phi(\underbrace{x - ty}_{z})\, dx \right] dt$$

$\Omega \leftarrow \Omega_{\delta/2}$

Since $\forall x \in \Omega_{\delta/2}^c \Rightarrow \phi(x - ty) = 0$

$$= \int_0^1 \int_{\Omega_\delta} \sum y_j \, g_j(z + ty)\, \phi(z)\, dz\, dt$$

$\forall \psi \quad \int_{B_{\frac{\delta}{2}}} \psi = 1 \quad \& \quad \operatorname{supp} \psi \subset B(0, \delta/2)$

$$\Rightarrow \int_{B_{\frac{\delta}{2}}} \underbrace{\psi(y)\, T(\phi_y)}\, dy - T(\phi) = \int_{B_{\frac{\delta}{2}}} \psi(y) \left(T(\phi_y) - T(\phi) \right) dy$$

$$= \int_{\Omega_\delta} \left[\int_0^1 \int_{B_{\frac{\delta}{2}}} \psi(y) \sum y_j \, g_j(z + ty)\, dy\, dt \right] \phi(z)\, dz$$

$$\Rightarrow T(\phi) = \int_{B_{\frac{\delta}{2}}} \psi(y)\, T(\phi_y) - \text{RHS}.$$

But $\displaystyle\int_{B_{\frac{\delta}{2}}} \psi(y)\, T(\phi_y) = T(\underbrace{\psi * \phi}) = \phi * \psi$

$$= \int_{\Omega_\delta} \phi(x)\, T(\psi_x)\, dx$$

Hence if let

$$f(x) = T(\psi_x) - \sum_{j=1}^{n} \int_{B_{\delta/2}} \psi(y) \int_0^1 y_j\, g_j(x+ty)\, dt\, dy$$

We have $\displaystyle T(\phi) = \int f\, \phi \qquad \forall\, \phi \in C_c^\infty(\Omega_\delta)$.

This construct $f \in C(\Omega_\delta)$

On the other hand $\partial_j f = g_j \in C^0(\Omega)$ (or $L^1_{loc}(\Omega)$)

$\Rightarrow f \in W^{1,1}_{loc}(\Omega)$.

&

$$f(x+y) - f(x) = \int_0^1 \sum g_j(x+ty)\, y_j\, dt \quad \text{by } (1)$$

$\Rightarrow f$ is C^1

$\square$

<u>Corollary</u>: If T has $\partial_j T = 0 \Rightarrow$

$T(\phi) = c \int \phi$

(Exercise FD. 9.13).

(3) <u>Thm 6.14</u>: $\mathcal{N}_T := \{\phi \in \mathcal{D}(\Omega) \mid T(\phi) = 0\}$

Namely the kernel of T.

Let $S_1, \dots S_\ell \in \mathcal{D}'(\Omega)$. $T \in \mathcal{D}'(\Omega)$.

Assume $T(\phi) = 0$. $\forall \phi \in \bigcap_{i=1}^{\ell} N_{S_i}$

Then $\exists\ c_1, \dots c_\ell$ such that

$$T = \sum_{i=1}^{\ell} c_i S_i$$

Pf: W.L.G assume $\{S_i\}$ linearly indep.

$$V_S := \left\{ \begin{pmatrix} S_1(\phi) \\ \vdots \\ S_\ell(\phi) \end{pmatrix} \quad \forall \phi \in \mathcal{D}(\Omega) \right\}$$

V_S is a linear space.

Claim it is of dim ℓ.

If $\exists\ \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_\ell \end{pmatrix}$ $\quad \sum a_i S_i(\phi) = 0 \quad \forall \phi$

$\Rightarrow \quad \sum a_i S_i = 0 \Rightarrow (\Rightarrow \Leftarrow)$.

$$\Rightarrow \begin{pmatrix} S_1(\phi_1) \\ \vdots \\ S_\ell(\phi_1) \end{pmatrix} = \vec{v}_1 \quad \dots \quad \vec{v}_\ell = \begin{pmatrix} S_1(\phi_\ell) \\ \vdots \\ S_\ell(\phi_\ell) \end{pmatrix}$$

Form a basis of V_S.

$$M = \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_\ell \end{pmatrix}$$

$$\lambda_i(\phi) := \sum (M^{-1})_{ij} S_j(\phi)$$

Define $\quad \phi = v + \sum \lambda_i(\phi)\phi_i \quad$ (Namely $v \doteq \phi - \sum \lambda_i(\phi)\phi_i$)

Check $\quad v \in \bigcap N_{S_i}$

$$S_k(v) = S_k(\phi) - \sum \lambda_i(\phi) S_k(\phi_i) = 0$$

$$\left[\sum_i \lambda_i(\phi) S_k(\phi_i) = \sum (M^{-1})_{ij} S_j(\phi) \underbrace{S_k(\phi_i)}_{M_{ki}} = \delta_{jk} S_j(\phi) = S_k(\phi) \right]$$

Hence

$$T(\phi) = \sum \underbrace{(M^{-1})_{ij} \; T(\phi_i)}_{= c_j} S_j(\phi)$$

$\square$