① Three consequences

Thm 1: $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. $\mathcal{X}$ $\mathcal{Y}$ Banach

If T is bijective $\Rightarrow$ $T^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$.

Pf: T^{-1} exists. linear.

The continuity of T^{-1} $\Leftrightarrow$ $\forall\, U$ open $T(U)$ is open $\square$

Thm 2: If $\mathcal{X}$ $\mathcal{Y}$ Banach & $T: \mathcal{X} \to \mathcal{Y}$ has

$$G(T) = \{ (x, Tx) \subset \mathcal{X} \times \mathcal{Y} \}$$ closed, then

T is Continuous.

Pf: $\pi_1: G(T) \longrightarrow \mathcal{X}$ $\quad \pi_1(x, Tx) = x$

$\pi_2: G(T) \longrightarrow \mathcal{Y}$ $\quad \pi_2(x, Tx) = Tx$

$G(T) \subset \mathcal{X} \times \mathcal{Y}$ is a Banach space by Closedness.

$\Rightarrow$ π_1^{-1} is continuous. $\Rightarrow$

$$\mathcal{X} \xrightarrow{\pi_1^{-1}} G(T) \xrightarrow{\pi_2} \mathcal{Y}$$

is continuous.

$\pi_2: \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ is clearly continuous

$$\| \pi_2((x, y)) \| = \|y\| \leq \|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$$

Thm 3: $\mathcal{X}$ Banach $\mathcal{Y}$ - normed v.s.

$\mathcal{A} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

. If $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ $\forall x \in X$, then $\sup_{T \in \mathcal{A}} \|T\| < +\infty$

pf: $E_n := \bigcap_{T \in \mathcal{A}} \{x \mid \|Tx\| \leq n\}$

E_n is closed.

$\bigcup_{n=1}^{\infty} E_n = X \Rightarrow$ E_n can NOT be all nowhere dense

$\Rightarrow (\overline{E_n})^{\circ} \supset B(x_0, r) \Rightarrow \overline{B(x_0, r)} \subset E_n$

$\overline{E_n} = E_n$

$\Rightarrow$

$\|T(x_0)\| \leq n \quad \forall T \in \mathcal{A}$

& $\forall x \in \overline{B(0, r)}$ $\quad x + x_0 \in B(x_0, r)$

$\|T(x + x_0)\| \leq n \Rightarrow \|T(x)\| \leq \|T(x + x_0) - T(x_0)\|$

$\leq \|T(x + x_0)\| + \|T(x_0)\| \leq 2n$

Hence $\overline{B(0, r)} \subset E_{2n} \Rightarrow$

$\sup_{T \in \mathcal{A}} \sup_{\|x\| = r} \frac{\|Tx\|}{\|x\|} \leq \frac{2n}{r} \Rightarrow \sup_{T \in \mathcal{A}} \|T\| \leq \frac{2n}{r}$. $\square$

Simple Consequences: $\{f_n\}$ a sequence of $L^p(X)$.

Assume that $\forall \ell \in (L^p(X))^*$ $\{\ell(f_n)\}$ is bounded.

Then $\{\|f_n\|_{L^p}\}$ is bounded. $(1 < p < \infty)$.

$\ell = \phi_g \Rightarrow \quad \ell(f_n) = \phi_g(f_n) = \phi_{f_n}(g)$

$|\ell(f_n)| = |\phi_{f_n}(g)| < \infty$

$\Rightarrow \quad \|\phi_{f_n}\| = \|f_n\|_{L^p} < +\infty$ by Thm 3.

For $p=1$. define $\phi_{f_n}(g) := \int f_n g \quad \forall g \in L^\infty(X)$

$|\phi_{f_n}(g)| \leq \|f_n\|_1 \|g\|_{L^\infty}$

Hence $\forall \ell \in (L^1(X))^*$ $\{\ell(f_n)\}$ is bounded implies that

$|\phi_{f_n}(g)|$ is bounded $\forall g \in L^\infty(X)$ $\Longrightarrow$ $\{\|\phi_{f_n}\|\}$ is bounded.

$\Rightarrow$ $\|f_n\|_1$ is bounded.

(Prop 6.13)

($p=\infty$, works similarly. But require μ is semi-finite)

(2) Remarks on $(L^\infty)^*$.

$X = [0,1] \quad \mu = dm$.

$\phi: f \to f(0), \quad \forall f \in C([0,1])$.

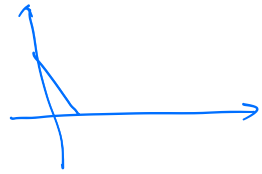
$C([0,1]) \subset L^\infty([0,1])$ is a linear subspace.

$\phi|_{C([0,1])}$ is a bounded linear functional.

$\Rightarrow \phi$ can be extended to $L^\infty([0,1])$ as a bounded linear functional.

Claim: $\nexists\, g \in L^1$ such that $\phi(f) = \int_0^1 g f$

Consider $f_n = \max\{1-nx, 0\}$



$\phi(f_n) = 1$

$\int_0^1 g f_n \to 0$ as $n \to \infty$ by DCT

$1 \neq 0 \Rightarrow$ Such $\underset{\in L^1}{g}$ can NOT exist.

③ Weak topology & weak *- topology.

Weak topology: $\mathcal{X}$ a normed vector space.

$\mathcal{X}^*$ its dual. The weak topology is generated by $f \in \mathcal{X}^*$

Namely $U_{i_1 \dots i_r} := \{x \mid |f_{i_1}(x)| < \varepsilon_{i_1} \quad \dots \quad |f_{i_r}(x)| < \varepsilon_{i_r}\}$

are $\underset{\text{basis}}{\text{neighborhood}}$ of $\{0\} \subset \mathcal{X}$.

$\forall x_0 \in \mathcal{X}$ $\quad \{x_0 + U_{i_1 \dots i_r}\}$ forms neighborhood basis there.

Weak*- topology on $\mathcal{X}^*$: $\quad \mathcal{X}^*$ has a weak topology (as a normed vector space)

But it has another topology:

$$V_{i_1 \dots i_r} := \{\underset{\in \mathcal{X}^*}{f} \mid |\hat{x}_{i_1}(f)| < \varepsilon_1 \quad \dots \quad |\hat{x}_{i_r}(f)| < \varepsilon_{i_r}\}$$

generateds the neighborhood near $0 \in \mathcal{X}^*$

<u>Theorem 1</u>: $\{x_n\} \to x$ w.r.t the weak topology on $\mathcal{X}$

iff $\forall f \in \mathcal{X}^*$ $\{f(x_n)\} \to f(x)$.

<u>Theorem 2</u>. If $\{x_n\}$ is weakly convergent $\Longrightarrow$ $\{x_n\}$ is bounded

Thm 3: Let $\{f_n\}$ be a weak* convergent sequence

$\mathcal{X}$ is complete $\Longrightarrow$ $\{f_n\}$ is bounded.

Assuming Theorem 1. (& its corresponding result for weak*)

Thm 2 & Thm 3 are immediate consequences of UBP.