0 Evolution of topology:

(I) Euclidean topology

$\left(\mathbb{R}^n, d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\right)$ $\longrightarrow$ (II) (X, d)

metric topology

open set is defined by $B(x_0, r)$

$\longrightarrow (X, \mathcal{T})$ — general topology.

(III).

Baby Rudin did II & I. We shall move to (II) & (III)

Baire Category Theorem is a result of II.

[Covered in Rudin's Principle ch3 Exercise].

This is a metric topology result!

Why we need general topology?

Consider $X = [0,1]$. X^X is the maps: $[0,1] \to [0,1]$.

$\{f_n\} \to f$ p.w. $f_n(x) \to f(x)$.

iff $\{f_n\} \to f$ w.r.t the product topology. [Prop 4.12]

The topology on X^X is the so-called product topology (See §4.2).

This topology can not be given via a metric.

① Baire theorem:

Let X be a complete metric space.

ⓐ If $\{U_n\}$ is a sequence of open dense subset of X then $\bigcap_{n=1}^{\infty} U_n$ is dense in X

ⓑ X is NOT a countable union of nowhere dense sets.

A is dense if $\bar{A} = X$

A is nowhere dense if $\bar{A}^\circ = \phi$.

E.g. $U_n = \mathbb{R}^n \setminus \{x_1, x_2 \cdots x_n\}$

<u>Pf.</u>: First ⓑ follows from ⓐ.

If $X = \bigcup E_n$ $\quad (\bar{E}_n)^\circ = \phi \Rightarrow (\bar{E}_n)^c$ is dense.

Otherwise $\exists B(x_0, \delta) \cap (\bar{E}_n)^c = \phi \Rightarrow B(x_0, \delta) \subset \bar{E}_n \Rightarrow$

$B(x_0, \delta) \subset (\bar{E}_n)^\circ$. A contradiction!

Then $X = \bigcup E_n \Rightarrow \phi = \bigcap E_n^c$

$\Rightarrow$ by $(\bar{E}_n)^c \subset E_n^c$ (since $E_n \subset \bar{E}_n$)

$\bigcap (\bar{E}_n)^c = \phi$ But this contradicts ⓐ

For ⓐ, we observe the following lemma.

If $\overline{B(x_n, r_n)} \subset \overline{B(x_{n-1}, r_{n-1})}$ are nested closed balls

& $r_n \to 0$, then $\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \neq \phi$.

<u>Pf</u>: $d(x_n, x_{n+m}) \leq r_n \implies \{x_n\}$ is Cauchy

Hence $\{x_n\} \to x$

By $x_{n+m} \in \overline{B(x_n, r_n)} \implies x \in \overline{B(x_n, r_n)} \quad \forall n$

$\implies x \in \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}$

Now, we show $\forall W \neq \phi$. open $W \cap \bigcap_{n=1}^{\infty} U_n \neq \phi$

This $\implies \bigcap_{n=1}^{\infty} U_n$ is dense.

$U_1 \cap W$ is open, since U_1 is dense $\implies U_1 \cap W \neq \phi$

$\implies \overline{B(x_1, r_1)} \subset U_1 \cap W$. $\quad r_1 \leq \frac{1}{2}$

Now we consider $B(x_1, r_1) \cap U_2$ — open & $\neq \phi$

$\implies \exists \ \overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap U_2 \quad r_2 \leq \frac{1}{2^2}$

$\vdots$

$\overline{B(x_j, r_j)} \subset B(x_{j-1}, r_{j-1}) \cap U_j \quad r_j \leq \frac{1}{2^j}$

Hence we have $\{\overline{B(x_j, r_j)}\}$ nested balls.

$\overline{B(x_1, r_1)} \subset U_1 \cap W$

Lemma $\implies \bigcap_{j=1}^{\infty} \overline{B(x_j, r_j)} \neq \phi$

$\overline{B(x_j, r_j)} \subset \left(\bigcap_{k=1}^{j} U_k\right) \cap W$

$\implies \exists \ x \in \left(\bigcap_{k=1}^{\infty} U_k\right) \cap W$. $\square$

② Open mapping theorem: If $T: \mathcal{X} \to \mathcal{Y}$ is a surjective bounded linear map from $\mathcal{X}$ to $\mathcal{Y}$ with $\mathcal{X}$ & $\mathcal{Y}$ are Banach, then T is an open map.

<u>I.e.</u> $\forall\ U$ open $T(U) \subset \mathcal{Y}$ is open.

<u>Pf</u>: Reduction: suffices to show $T(B(0,1)) \supset B(0, \delta)$ for some $\delta > 0$.

To prove $T(U)$ is open, $\forall\ y_0 = T(x_0)$ $y_0 \in T(U)$. $x_0 \in U$.

$$\Rightarrow \quad B(x_0, \lambda) \subset U \Longrightarrow T(B(x_0, \lambda)) = T(x_0) + T(B(0, \lambda))$$

$$= \underset{\underset{y_0}{\parallel}}{T(x_0)} + \lambda T(B(0,1)) \supset y_0 + \lambda B(0, \delta) = B(y_0, \lambda\delta)$$

Namely $B(y_0, \lambda\delta) \subset T(U)$. □

<u>Step 1</u>: By Baire's theorem, $\exists\ r > 0$ $\overline{T(B(0,1))}$ is dense in $B(0, r)$.

<u>Step 2</u>: $T(B(0,2)) \supset B(0, r)$. $\left[\Rightarrow T(B(0,1)) \supset B(0, \frac{r}{2})\right]$.

<u>pf of Step 1</u>: $\displaystyle\bigcup_{n=1}^{\infty} \overline{T(B(0,n))} = \mathcal{Y}$

$\Rightarrow \exists\ n_1$, $T(B(0, n_1))$ is dense in a ball $B(y_0, r_1)$

$\Rightarrow \exists\ x_0$ $T(x_0) = y_0 \Rightarrow$

$T(\underbrace{B(0, n_1) - x_0}_{\subset B(0, n_1 + |x_0|)})$ is dense in $B(0, r_1)$

$\Rightarrow \quad T(B(0, n_1 + |x_0|))$ is dense in $B(0, r_1)$

$\Rightarrow \quad T(B(0,\lambda_1))$ is dense in $B(0, r_1)$

$\Rightarrow \quad T(B(0,1))$ is dense in $B(0, \frac{r_1}{\lambda_1}) := B(0, r)$

Now $\quad T(B(0,c))$ is dense in $B(0, cr)$. $\qquad (*)$

For step 2. $\quad \forall u \in B(0,r)$. By $(*)$

$\Rightarrow$ $\exists x_1 \in B(0,1)$, $\quad |u - Tx_1| < \frac{r}{2}$

$\overset{(*)}{\Rightarrow} \quad \exists x_2 \in B(0, \frac{1}{2}) \qquad |u - Tx_1 - Tx_2| < \frac{r}{4}$

inductively $\Rightarrow \quad \exists x_n \in B(0, \frac{1}{2^{n-1}})$

Since $\quad |u - Tx_1 - \cdots Tx_{n-1}| < \frac{r}{2^{n-1}}$, by $(*)$

$\exists \; x_n \in B(0, \frac{1}{2^{n-1}})$ Such that

$$|u - Tx_1 - \cdots - Tx_{n-1} - Tx_n| < \frac{r}{2^n}$$

Now let $\quad x = \sum x_i$

$\sum |x_i| \leq \sum \frac{1}{2^{i-1}} < \infty \Rightarrow \quad x \in X$ by completeness

$|u - \sum_{i=1}^{\infty} Tx_i| = 0 \quad \Rightarrow \quad u = T(\sum_{i=1}^{\infty} x_i) = T(x)$.

$x \in B(0, 2)$.