① $T \in \mathcal{D}'(\Omega)$. $\underline{\psi \in C_c^\infty(\mathbb{R}^n)}$

$(T * \psi)(x)$ is defined for $x \in V$

$$V := \{ x \mid x - \mathrm{supp}(\psi) \subset \Omega \}$$

It is defined as

$$T(\tau_x \tilde{\psi}) \qquad \tilde{\psi}(y) = \psi(-y)$$

$$(\tau_x \tilde{\psi})(y) = \tilde{\psi}(y - x) = \psi(x - y)$$

Hence if $x \in V$, $\mathrm{supp}(\tau_x \tilde{\psi}) \subset \Omega$

$$\mathrm{supp}(\tau_x \tilde{\psi}) = x - \mathrm{supp}\,\psi$$

Hence $(T * \psi)(x) := T(\tau_x \tilde{\psi})$ makes sense.

<u>Main result:</u> (Prop 9.3).

ⓐ $T * \psi \in C^\infty(V)$

ⓑ $\partial^\alpha (T * \psi) = (\partial^\alpha T) * \psi = T * (\partial^\alpha \psi)$

ⓒ $\forall \phi \in C_c^\infty(V)$, $\int (T * \psi)\phi = T(\phi * \tilde{\psi})$

<u>Lemma</u> (LL. Lemma 6.8). $\mathcal{O}_\phi := \{ x \mid x + \mathrm{supp}\,\phi \subset \Omega \}$

$\mathcal{O}_\phi$ is open (nonempty if $\mathrm{supp}\,\phi \subset \Omega$).

$y \to T(\phi_y)$ is C^∞ $\quad \phi_y(x) \doteq \phi(x - y)$.

(a) $D_y^\alpha T(\phi_y) = (-1)^{|\alpha|} T((D^\alpha \phi)_y) = (D^\alpha T)(\phi_y)$ (1)

(b) $\forall \varphi \in C_c^\infty(O_\phi)$

$$\int_{O_\phi} \varphi(y)\, T(\phi_y)\, dy = T(\varphi * \phi) \qquad (2)$$

<u>Assume Lemma</u>: Apply it to $\phi = \tilde{\psi}$

$D_y^\alpha T(\phi_y) = D^\alpha (T * \psi)$

$\|$ Lemma part (a)

$D^\alpha T(\phi_y) = (D^\alpha T) * \psi$

Lemma part (a)

$D_y^\alpha T(\phi_y) = (-1)^{|\alpha|} T((D^\alpha \phi)_y)$

$\boxed{D^\alpha \phi = D^\alpha \tilde{\psi} = (-1)^{|\alpha|} \widetilde{(D^\alpha \psi)}}$

$\Rightarrow D_y^\alpha T(\phi_y) = T\left(\widetilde{(D^\alpha \psi)}_y\right)$

$= T * D^\alpha \psi$

This proves (a) & (b)

For (c) (2) of lemma $\Rightarrow \displaystyle\int_V \varphi(y)\, \underbrace{T(\tau_y \tilde{\psi})}_{(T*\psi)(y)} = T(\varphi * \tilde{\psi})$

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(2) Proof of the Lemma

Continuity $\phi_y(x) = \phi(x-y)$

$\phi(x-y) \to \phi(x)$ in $\mathcal{D}(\Omega)$ if $y \to 0$

$\Rightarrow T(\phi_y)$ is continuous

$$\frac{1}{h}\left[\phi(x-y-he_j)-\phi(x-y)\right]-(-1)\partial_j\phi(x-y)\xrightarrow[\text{as } h\to 0]{}0 \quad \text{in } \mathcal{D}(\Omega)$$

Since $\frac{1}{h}\left[D^\alpha\phi(x-y-he_j)-D^\alpha\phi(x-y)\right]-(-1)\partial_j D^\alpha\phi(x-y)$

$$\longrightarrow 0$$

Hence

$$\frac{1}{h}\left[T(\phi(x-y-he_j))-T(\phi(x-y))\right]-(-1)T(\partial_j\phi(x-y))$$

$$\longrightarrow 0$$

Namely $\partial_j T(\phi_y) - (-1)T((\partial_j\phi)_y) = 0$

$$\partial_j(T(\phi_y)) = (-1)T((\partial_j\phi)_y) = (\partial_j T)(\phi_y).$$

$$(\partial_j\phi)_y = \partial_j^x(\phi_y)$$

$\Rightarrow (-1)T(\partial_j^x(\phi_y)) = (\partial_j T)(\phi_y)$. This proves $|\alpha|=1$.

By induction ① holds.

Now we show ②.

The key observation

$$\varphi*\phi(x) = \int_{O_\phi}\phi(x-y)\varphi(y)\,dy$$

$$= \lim_{\text{diam}(A_k)\to 0}\sum\frac{1}{|A_k|}\phi(x-y_k)\varphi(y_k) \qquad y_k\in A_k\subset O_\phi$$

&

$$D^\alpha(\varphi*\phi) = \int_{O_\phi}D^\alpha\phi(x-y)\varphi(y)\,dy$$

$$= \lim_{\operatorname{diam}(A_k)\to 0} \sum \frac{1}{|A_k|} D^{\alpha}\phi(x-y_k)\,\varphi(y_k)$$

$K = \operatorname{supp}\varphi + \operatorname{supp}\phi \subset\subset \Omega$

Namely $\sum \frac{1}{|A_k|} \phi(x-y_k)\,\varphi(y_k) \longrightarrow \varphi * \phi$ in $\mathcal{D}(\Omega)$

$$\Rightarrow \lim_{\operatorname{diam}(A_k)\to 0} T\left(\sum \frac{1}{|A_k|}\phi(x-y_k)\,\varphi(y_k)\right) \longrightarrow T(\varphi * \phi)$$

$$\| $$

$$\int_{U_\phi} T(\phi_y)\cdot\varphi(y) = T(\varphi * \phi)$$

$\square$

③ Remarks: ① If $T = T_f$

$$(T * \psi)(\phi) = \int \left(\int f(\underset{\overset{\|}{z}}{x-y})\,\psi(y)\,dy\right)\phi(x)\,dx$$

$$= \int\left(\int f(z)\,\psi(x-z)\,dz\right)\phi(x)\,dx = \int f(z).\,(\phi * \tilde{\psi})(z)$$

Also $(T_f * \psi)(x) := \int f(x-y)\psi(y)$

$= \int f(z)\,\psi(x-z)$

$= \int f(z)\,(\tau_x\tilde{\psi})(z)$

$$\left[\begin{aligned}(\phi * \tilde{\psi})(z) &= \int \phi(z-y)\,\tilde{\psi}(y)\,dy \\ &= \int \phi(\underbrace{z-y}_{\overset{\|}{x}})\,\psi(-y)\,dy \qquad -y = x-z \\ &= \int \phi(x)\,\psi(x-z)\,dx\end{aligned}\right]$$

Namely ② holds.

(b) By the definition $(T*\psi)(y) = T(\tau_y \tilde{\psi})$

$\Rightarrow$ For $(\delta_0 * \psi)(y) = \delta_0(\tau_y \tilde{\psi})$

$= \delta_0(\psi(y-x)) = \psi(y)$

$\Rightarrow \quad \delta_0 * \psi = \psi \qquad \forall \psi \in C_c^\infty(V).$

Namely δ_0 is an identity in terms of $*$

(4) <u>Prop 9.5</u>: $\Omega \subset \mathbb{R}^n$. $C_c^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$.

Two steps.

Use the standard cut-off function $j(x) \in C_c^\infty(\mathbb{R}^n)$

$\operatorname{supp} j = \overline{B(0,1)}$ & $\int j = 1$ & $j \geq 0$, $j(x) = j(|x|)$

Consider $j_\varepsilon * T$, if $\operatorname{supp} T \subset\subset \Omega$ then $j_\varepsilon * T$ has compact support. $\varepsilon \ll 1$

$\operatorname{supp} \varphi \subset \Omega \setminus \operatorname{supp} T$

$\int j_\varepsilon * T(x)\, \varphi(x) = T(j_\varepsilon * \varphi) = 0$

For general $T \in \mathcal{D}'(\Omega)$. pick $\overline{\Omega_k}$ compact $\bigcup \Omega_k = \Omega$.

$\Omega_k \subset \Omega$

$\Omega_k := \{x \in \Omega \mid d(x, \Omega^c) > \frac{1}{k}\}$

& ζ_k with $\zeta_k \equiv 1$ on $\overline{\Omega_k}$

& $\operatorname{supp} \zeta_k \subset\subset \Omega$.

$\Omega_{k+1} \supset \Omega_k$

$\zeta_k T \longrightarrow T$

Since $\forall \phi \in C_c^\infty(\Omega) \Longrightarrow \operatorname{supp} \phi \subset \overline{\Omega_k}$ for all $k \gg 1$

$\Rightarrow (\zeta_k T)(\phi) = T(\zeta_k \phi) = T(\phi)$

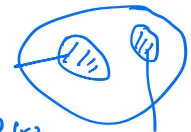
Namely $\lim_{k\to\infty} (\zeta_k T)(\phi) = T(\phi) \qquad \forall \phi \in C_c^\infty(\Omega).$

But $\quad j_\varepsilon * T \longrightarrow T \quad$ as element in $\mathcal{D}'(\Omega)$.

Since
$$(j_\varepsilon * T)(\phi) = T\left(\int j_\varepsilon(x-y)\phi(y)\right)$$
$$= T(\phi(x))$$

Here $\operatorname{supp} T \subset\subset \Omega$

$\Rightarrow$ $\operatorname{supp} T$



$\tilde{j}_\varepsilon * \varphi(x)$

$= j_\varepsilon * \varphi(x)$

$= \int_{B_\varepsilon(0)} \underline{j_\varepsilon(z)}\,\varphi(x-z)\,dz = 0$

Since $\quad \int j_\varepsilon(x-y)\phi(y) \longrightarrow \phi(x) \quad$ uniformly

$$D^\alpha\left(\int j_\varepsilon(x-y)\phi(y)\right) = D^\alpha \int j_\varepsilon(y)\phi(x-y) = \int j_\varepsilon(y)(D^\alpha\phi)(x-y) \longrightarrow D^\alpha\phi(x).$$

Also uniformly

Namely
$$\int j_\varepsilon(x-y)\phi(y) \longrightarrow \phi(x) \quad \text{in } \mathcal{D}(\Omega)$$

Putting them together $\Rightarrow$ $\forall\, T \quad \exists \quad j_{\varepsilon_k} * \zeta_k T \in C_c^\infty(\Omega)$

such that $\quad j_{\varepsilon_k} * (\zeta_k T) \longrightarrow T \quad$ in $\mathcal{D}'(\Omega)$.

Namely any distribution is the limit of a sequence of $C_c^\infty(\Omega)$ functions.

where V comes from

$$V := \{ x \mid x - \operatorname{supp}\psi \subset \Omega \}$$

(i) $x - \operatorname{supp}\psi$ is compact $\Rightarrow$ V is open

(ii) $T = T_f$ or simply f, with $f \in L^1_{loc}(\Omega)$

Namely we only know $\int_K |f| < +\infty$, if $K \subset\subset \Omega$.

$$f * \psi(x) = \int f(z)\,\psi(x-z)\,dz$$

need to be finite, hence we want z to be in a compact sub set of Ω, when $\psi(x-z) \neq 0$

namely $x - z \in \operatorname{supp}\psi$ & $z \in \Omega$

This is the requirement $\underbrace{x - \operatorname{supp}\psi}_{\overset{\shortparallel}{K}} \subset \Omega$. comes in.

If so then,

$$\int_{\operatorname{supp}\psi} f(\underset{\overset{\shortparallel}{z}}{x-y})\psi(y) = \int_{\underbrace{x-\operatorname{supp}(\psi)}_{\text{Compact } \subset \Omega}} f(z)\,\psi(x-z)\,dz \quad \text{something finite}$$

$\underline{y = x - z}$