1 Some background / motivations

a Solution to a PDE

(i) $U_{xx} - U_{yy} = 0$

$$U(x, y) = f(x+y) + g(x-y)$$

Extending the concept of the solution to make sense for f, g are only continuous. [Viscosity solutions]

(ii) $\begin{cases} U_{xx} + U_{yy} = 0 \\ u|_{\partial\Omega} = \varphi \end{cases}$ used by Riemann but has a hole.

Dirichlet principle. ~ Extending the scope of the solution effectively reduce finding a solution into two steps: A: Find a weak solution

B: Prove the weak solution is regular

b Fourier Transform: $\langle x, \xi \rangle$ — pairing, which is bilinear

$(\, , \,)$ — the Hermitian product.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \, x \cdot \xi} f(x)\, dx$$

$$\widehat{f'} = \int_{-\infty}^{\infty} e^{-2\pi i \, x\cdot\xi} f'(x)\, dx = 2\pi i \xi \int_{-\infty}^{\infty} e^{-2\pi i \, x\cdot\xi} f(x)$$

$$= 2\pi i \xi \, \hat{f}$$

Derivative in x-space $\leftrightarrow$ multiplication in ξ-space

It is desirable to have more inclusive scope for derivative

Ⓒ Weak derivatives

$$f \in L^1_{loc}(\Omega)$$

$g := \frac{\partial f}{\partial x_i} \in L^1_{loc}(\Omega)$ is called a weak derivative

if $\forall\, \varphi \in C_c^\infty(\Omega)$ $\displaystyle\int_\Omega \frac{\partial f}{\partial x_i}\varphi = -\int_\Omega f \frac{\partial \varphi}{\partial x_i}$

$$g * \phi_t = \int g(y)\,\phi_t(x-y) = -\int f(y)\frac{\partial \phi_t}{\partial y_i} \xrightarrow{a.e.} g \quad \text{as } t \to 0$$

$\Rightarrow$ g is decided a.e (by our approximation result in Week 1).

$$W^{1,p}_{loc}(\Omega) := \left\{ f \in L^p_{loc},\ \frac{\partial f}{\partial x_i} \in L^p_{loc} \right\}$$

$$W^{1,p}(\Omega) := \left\{ f \in L^p,\ \frac{\partial f}{\partial x_i} \in L^p \right\}.$$

These are the so-called Sobolev spaces.

② Definitions.

$\mathcal{D}(\Omega)$ — $\{ f,\ f \in C_c^\infty(\Omega) \}$

topology: $\varphi_j \in \mathcal{D}(\Omega)$ $\quad \varphi_j \to \varphi$

iff $\exists K \subset\subset \Omega$ $\quad \operatorname{supp}\varphi, \operatorname{supp}\varphi_j \subset K$

& $\quad D^\alpha \varphi_j \to D^\alpha \varphi$ uniformly on K, $\forall \alpha$

$\mathcal{D}'(\Omega)$ — the space of distributions is defined as the collection of linear maps: $T: \mathcal{D}(\Omega) \to \mathbb{C}$, satisfying

if $\varphi_j \to \varphi$ in $\mathcal{D}(\Omega)$

$T(\varphi_j) \to T(\varphi)$.

Topology: $T^j \to T$ in $\mathcal{D}'(\Omega)$: if $\forall \varphi \in C_c^\infty(\Omega)$.

$T^j(\varphi) \to T(\varphi)$. $\quad$ (Some kind of weak*-topo)

3. Examples.

a. $f \in L^1_{loc}(\Omega)$ $\quad (L^p_{loc} \subset L^1_{loc})$

means $\forall K \subset\subset \Omega$ (compact) $\quad \int_K |f| < +\infty$

$\left(\int_K |f| \le \|f\|_p \, K^{\frac{1}{p'}} < +\infty \text{ if } f \in L^p_{loc} \right)$.

$$T_f(\varphi) = \int f \varphi$$

$\Rightarrow$ If $\varphi_j \to \varphi$ then $T_f(\varphi_j) - T_f(\varphi) = \int_K f(\varphi_j - \varphi)$

$\le \sup|\varphi_j - \varphi| \int_K |f| \to 0$ as $j \to \infty$

(b) μ - A Radon measure

$$T_\mu(\varphi) := \int \varphi \, d\mu$$

$\varphi_j \to \varphi$ $\quad \left| T_\mu(\varphi_j) - T(\varphi) \right| = \left| \int (\varphi_j - \varphi) \, d\mu \right|$

$$\leq \left(\int_K d\mu \right) \left| \varphi_j - \varphi \right|_{L^\infty(K)} \to 0$$

In particular, $\delta_p(\varphi) := \varphi(p)$
is a distribution.

(c) $T: \varphi \to D^\alpha \varphi(p)$ is also a distribution.

$p \in \Omega, \quad \forall \alpha.$ $\quad \left[= (-1)^{|\alpha|} D^\alpha \delta_p \right]$

(4) Two results via approximation.

<u>Prop 9.1</u>: $\quad f \in L^1(\mathbb{R}^n)$, $\int f = a$

$\forall t \quad f_t(x) = \frac{1}{t^n} f\left(\frac{x}{t}\right)$

Then $f_t \longrightarrow a \, \delta_0$ in $\mathcal{D}'$

Pf: $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$

$$T_{f_t}(\varphi) = \int f_t(x) \varphi(x) \, dx = (f_t * \varphi)(0)$$

$\longrightarrow a \varphi(0)$, by Theorem 8.14. since φ is uniformly bounded & continuous on $K \subset\subset \mathbb{R}^n$

<u>Theorem</u> (6.5 of LL). $f, g \in L^1_{loc}(\Omega)$.

If $T_f = T_g \Rightarrow f = g$ a.e.

<u>Pf</u>: Let φ be the $C_c^\infty(\mathbb{R}^n)$ $\operatorname{supp}\varphi \subset \overline{B(0,1)}$

$$f * \varphi_t(x) = \int_{B(0,t)} f(\underbrace{x-y}_{z})\, \varphi_t(y)$$

is well defined $\forall x \in \Omega_t := \{ \overset{\in\Omega}{z} \mid d(z, \partial\Omega) < t \}$

$$= \int_{\Omega \supset \overline{B(x,t)}} f(z)\, \varphi_t(x-z)\, dz$$

$$= T_f(\varphi_t(x-z))$$

By $T_f = T_g \Rightarrow f * \varphi_t(x) = g * \varphi_t(x) \quad \forall x \in \Omega_t$

Hence by the approximation theorem

$$f*\varphi_t(x) \to f(x), \quad \to g(x) \qquad \forall x, \text{ Lebesgue point of } f \,\&\, g$$

$$\Rightarrow \quad f(x) = g(x). \quad \text{a.e.}$$

(5) Derivatives & multiplication

$$(D^\alpha T)(\varphi) \doteq (-1)^{|\alpha|}\, T(D^\alpha \varphi)$$

Clearly linear

If $\varphi_j \to \varphi$ $\Rightarrow$ $\exists K \subset\subset \Omega$ $D^{\alpha+\beta}\varphi_j \longrightarrow D^{\alpha+\beta}\varphi$ uniformly on a K

$\operatorname{Supp}\varphi_j$, $\operatorname{supp}\varphi \subset K$

$\Rightarrow$ $(D^\alpha T)(\varphi_j) - (D^\alpha T)(\varphi)$

$= (-1)^{|\alpha|}\, T(D^\alpha\varphi_j - D^\alpha\varphi)$

Note $D^{\alpha+\beta}\varphi_j \longrightarrow D^{\alpha+\beta}\varphi$ $\forall\beta$ uniformly on K

$\Rightarrow$ $(D^\alpha T)(\varphi_j) \longrightarrow (D^\alpha T)(\varphi)$.

Claim: $D^\alpha : \mathcal{D}' \longrightarrow \mathcal{D}'$ is continuous, namely if $T_j \to T$

$D^\alpha T_j \longrightarrow D^\alpha T$

$(D^\alpha T_j)(\varphi) = (-1)^{|\alpha|}\, T_j(D^\alpha\varphi) \longrightarrow (-1)^{|\alpha|}\, T(D^\alpha\varphi) = (D^\alpha T)(\varphi)$

. $\forall \psi \in C^\infty(\Omega)$ $\psi\cdot T(\varphi) = T(\psi\cdot\varphi)$

Clearly $\psi\varphi \in C_c^\infty(\Omega)$

Again $\psi\cdot T \in \mathcal{D}'(\Omega)$ since if $\varphi_j \longrightarrow \varphi$ in $\mathcal{D}(\Omega)$

$\Rightarrow$ $\operatorname{Supp}\varphi_j$ $\operatorname{supp}\varphi \subset K$ $D^\alpha\varphi_j \to D^\alpha\varphi$ uniformly

But $\operatorname{Supp}\psi\varphi_j \subset K$ as well.

& $D^\alpha(\psi\varphi_j) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta\psi\, D^\gamma\varphi_j \longrightarrow \sum_{\beta+\gamma=\alpha} D^\beta\psi\, D^\gamma\varphi \left(\frac{\alpha!}{\beta!\gamma!}\right)$

$\qquad\qquad = D^\alpha(\psi\varphi)$.

The bigger of the space (& weaker of topology), the smaller is the dual

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$C^\infty(\Omega) \supset C_c^\infty(\Omega)$ (Pic 1)

$* \downarrow$

Distribution T with compact support $\subset \mathcal{D}'(\Omega)$

$L^1(\mathbb{R}^n)$ $\left(C^\infty(\mathbb{R}^n) \right) \supset \mathcal{S} \supset C_c^\infty(\mathbb{R}^n)$

$* \downarrow$

$L^\infty(\mathbb{R}^n)$ (Distribution with compact support) $\subset \mathcal{S}' \subset \mathcal{D}'(\mathbb{R}^n)$

$\uparrow$ tempered distribution

(Pic 2)