① Finish the proof of (*) – (hence Lebesgue Diff).

Pf of (*): $\displaystyle\lim_{r\to 0} ⨍_{B(x,r)} f(y)\,dy = f(x)$. a.e. (*)

idea: If f is continuous (*) is obvious.

Hence find g $\int |f-g| < \varepsilon$

Now $\displaystyle\varlimsup_{r\to 0} \left| ⨍_{B(x,r)} f(y) - f(x) \right|$

$\displaystyle\leq \varlimsup_{r\to 0} \left| ⨍_{B(x,r)} f(y)-g(y) + g(y)-g(x) + g(x)-f(x) \right|$

$\displaystyle\leq \underbrace{\varlimsup_{r\to 0} \left| ⨍_{B(x,r)} f(y)-g(y) \right|} + |g(x)-f(x)|$

$\forall \alpha>0$ let

$\displaystyle E_\alpha := \left\{ x \;\middle|\; \varlimsup_{r\to 0} \left| ⨍_{B(x,r)} f(y)-f(x) \right| > \alpha \right\}$

& $F_\alpha := \{ x \mid \boxed{} > \alpha \}$. $G_\alpha = \{ x \mid |g(x)-f(x)| > \alpha \}$

$\Rightarrow E_\alpha \subset F_{\frac{\alpha}{2}} \cup G_{\frac{\alpha}{2}}$

$\Rightarrow |E_\alpha| \leq |F_{\frac{\alpha}{2}}| + |G_{\frac{\alpha}{2}}|$

But by Lemma $\displaystyle |F_{\frac{\alpha}{2}}| \leq \frac{C}{\alpha} \int |f(y)-g(y)| \to 0$

$\displaystyle |G_{\frac{\alpha}{2}}| \leq \frac{C}{\alpha} \int |f(y)-g(y)| \to 0$ if g is chosen suitably. □

② Some notation & remarks.

<u>Lemma</u>: H: $f \in L^1 \longrightarrow Hf \in$ Weakly L^1

<u>Defn</u>: (i) $\lambda_g(\alpha) := \left|\left\{ x \mid |g(x)| > \alpha \right\}\right|$

or $\mu(\{|g(x)| > \alpha\})$ for g measurable of $(X, \mathcal{M}, \mu)$.

(ii) $[f]^p := \sup\limits_{\alpha > 0} \left\{ \left(\alpha^p \lambda_f(\alpha)\right)^{1/p} \right\}$

Trivial estimate:

$$\lambda_f(\alpha)\, \alpha^p \leq \int |f|^p \iff [f]_\alpha^p \leq \|f\|_p^p$$

Lemma on Maximal Function shows

$$[Hf] \leq C \|f\|_1$$

③ Thorin-Riesz Thm