1 $p=1$ case & the $1<p<\infty$ without σ-finiteness

Pf: The proof for $\mu(X)<\infty$ & $\mu(X)$ is σ-finite works for $p=1$. — Exercises.

However, the following argument does not work for $q=\infty$ $(p=1)$

Now we prove the result for μ in general, namely without σ-finiteness for $1<p<\infty$ ($\Rightarrow$ $1<q<\infty$)

$\forall F$ subset of X F is σ-finite $\Longrightarrow \exists\, g_F$

$$\phi(f)=\int g_F f \qquad \forall f\in L^p(F)$$

$$\|g_F\|_q\leq\|\phi\|$$

Define
$$M:=\sup_{F-\sigma\text{-finite}}\|g_F\|_q$$

If $F_1 \dots F_n \dots$ σ-finite $\Longrightarrow F=\bigcup_{n=1}^{\infty}F_n$ is also σ-finite

Hence if $\|g_{F_i}\|_q\to M \Longrightarrow$ $\boxed{\text{Now let } g=g_F}$

$M=\|g_F\|_q$ since clearly $\|g_{F_1}\|_q\leq\|g_{F_2}\|_q$ if $F_1\subset F_2$

Key observation: (i) $\forall A\supset F$ A is σ-finite set

$$g_A\big|_{A\setminus F}=0$$

The reason is $M^q\geq\int_A|g_A|^q=\int_F|g_F|^q+\int_{F\setminus A}|g_A|^q$

$\Longrightarrow g_A\big|_{A\setminus F}=0$ a.e. But fails if $q=\infty$. $\geq M^q$

(ii) $\forall f \in L^p$ $\quad A := F \cup \{ f(x) \neq 0 \}$

$\{ f(x) \neq 0 \} = \{ |f(x)| \neq 0 \} = \bigcup \{ \underbrace{x \mid |f(x)| \geq \tfrac{1}{n}}_{E_n} \}$

$m(E_n)\left(\frac{1}{n}\right)^p \leq \int_X |f|^p \leq C \Rightarrow m(E_n) \leq C n^p < \infty$

$\Rightarrow \phi(f) = \int_A f g_A$ since $f \in L^p(A)$

$= \int f g_A = \int f g_F = \int_X f g$ $\square$

(2) Lebesgue Differentiation Theorem.

Thm (3.20). $\forall f \in L^1_{loc}$

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| \, dm(y) = 0. \quad \text{a.e } x$$

Pf: Step 1: Reduce to show

$$\lim_{r \to 0} \fint_{B(x,r)} f(y) \, dm(y) = f(x). \quad \text{a.e. } x \qquad (*)$$

Assume $(*)$, let $Q \subset \mathbb{C}$ be $q_1 + i q_2$ $\quad q_i \in \mathbb{Q}$

Q is dense in $\mathbb{C}$

Apply $(*)$ to $|f(y) - q|$

$\Rightarrow \exists E_q \quad |E_q| = 0$

& $\forall x \notin E_q$

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y)-q| = |f(x)-q|$$

Now let $E = \bigcup E_q$ $\qquad |E| = 0$

$\forall x \notin E$, we have that

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y)-q| = |f(x)-q|$$

Hence $\displaystyle \overline{\lim_{r\to 0}} \fint_{B(x,r)} |f(y)-f(x)| \leq \overline{\lim_{r\to 0}} \fint_{B(x,r)} |f(y)-q| + |f(x)-q|$

$$= 2|f(x)-q|$$

On the other hand $\mathbb{Q}$ is dense, namely $\forall \varepsilon > 0\ \exists q$ $|f(x)-q| < \frac{\varepsilon}{2}$

$$\Rightarrow \quad \overline{\lim_{r\to 0}} \fint_{B(x,r)} |f(y)-f(x)| = 0$$

<u>Step 2</u>: Two lemmas.

① Covering lemma

Let $\mathcal{C}$ be a collection of open balls. & $U = \bigcup_{B\in\mathcal{C}} B$.

$\forall c < |U|$ $\exists$ disjoint $B_1 \cdots B_k \in \mathcal{C}$ such that

$$\sum_{j=1}^{k} |B_j| \geq \frac{c}{3^n}$$

Pf: $\forall K \subset U$, K compact, $|U| \geq |K| > c$

Pick finite subcover $\{B_j'\}_{j=1}^N$ covers K.

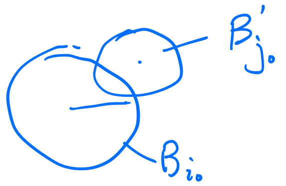
Now pick B_1 the biggest (radius)

B_2 the biggest among $\{B_j'\}_{j=1}^N$ - does not intersect B_1

$\cdots$ $\{B_1 \cdots B_k\}$

Claim. $\bigcup_{i=1}^k 3B_i \supset K$: $\forall y \in K$ $y \in B_{j_0}'$

But either $B_{j_0}' = B_{i_0}$ or B_{j_0}' intersects with B_{i_0}



$\Rightarrow$ $3B_{i_0} \supset B_{j_0}'$

$\Rightarrow$ the claim

$\Rightarrow$ $|K| \leq \sum_{i=1}^k 3^n |B_i|$ $\Rightarrow$ $\square$

(ii) The maximal function & its estimate.

Defn: $Hf(x) := \sup_{r>0} \fint_{B(x,r)} f(y)\, dy$

Lemma: $\exists C$

$$|\{Hf > \alpha\}| \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(y)|\, dy$$

Pf: $\left[\begin{array}{l}\text{Claim} \quad \int_{B(x,r)} f(y)\,dy \quad \text{is continuous in } (x,r). \\ \text{Check!}\end{array}\right]$

Let $E_\alpha := \{ |Hf > \alpha \}$, it is an open subset.

1st $\forall x \in E_\alpha$ $\exists r$ such that

$$\int_{B(x,r)} f(y)\,dy > \alpha\, |B(x,r)|$$

All such $\{B\}$ covers E_α.

By Lemma of Covering $\exists$ B_j such that

$\{B_j\}$ disjoint & $\sum |B_j| \geq \frac{c}{3^n}$ $\forall c < |E_\alpha|$

$$\Rightarrow \quad \frac{c}{3^n} \leq \sum |B_j| \leq \frac{1}{\alpha} \int_{B_j} f(y)\,dy$$

$$\Rightarrow \quad c \leq \frac{3^n}{\alpha} \|f\|_{L^1}$$

$c \to |E_\alpha| \Longrightarrow \square.$