$\exists$ exposition in HLP "inequalities". (slightly different)

$\exists$ Abstraction in Katznelson's "An Intro to Har Analysis"
(Formulated more abstractly)

It is well-known

$\log \|f\|_p$ is monotone increasing in p

& Convex in $\frac{1}{p}$

Namely $\quad \log \|f\|_{\frac{1}{\frac{t}{p_0}+\frac{1-t}{p_1}}} \leq t \log \|f\|_{p_0} + (1-t) \log \|f\|_{p_1}$

$\forall\, t \in [0, 1]$

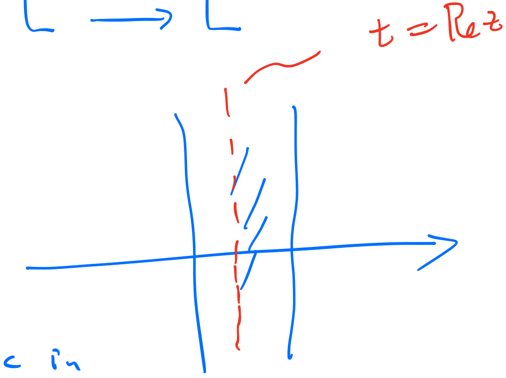
Riesz-Thorin theorem says this convexity also shows in the operator norms for $T:$ $\quad L^{p_0} \longrightarrow L^{q_0}$

$L^{p_1} \longrightarrow L^{q_1}$

key component:

Hadamard three line theorem:

$$|\phi(z)| \leq M_0^{1-t} M_1^{t}$$



$\forall \quad z = t + iy$, $\quad \phi$ holomorphic in the strip. $\quad M_0 = \sup\limits_{\mathrm{Re}\, z = 0} |\phi(z)| \qquad M_1 = \sup\limits_{\mathrm{Re}\, z = 1} |\phi(z)|$

Proof also uses Theorem 2.10. & 6.14 of Folland.

$$\|Tf\|_{q_t} = \sup_{\|g\|_{q_t'} = 1} \left| \int Tf \cdot g \right| \quad \leftarrow \text{Theorem 6.14}$$

Hence we reduce the theorem to show

$$\left\| \int Tf \cdot g \right\| \leq M_0^{1-t} M_1^{t} \qquad \forall f \in L^{p_t} \qquad (*)$$
$$g \in L^{q_t'}$$

with $\|f\|_{p_t} = 1 \qquad \|g\|_{q_t'} = 1$

Theorem 2.10 reduce $(*)$ to check it for step functions.

$$f = \sum c_j \chi_{E_j} \qquad g = \sum d_k \chi_{F_k}$$

$\{E_j\}$ $\{F_k\}$ disjoint.

$$\|f\|_{p_t} = \left(\sum |c_j|^{p_t} \chi_{E_j} \right)^{1/p_t}$$

$\mu(E_j)$

$$1 = \|f\|_{p_t} \iff \sum |c_j|^{p_t} |E_j| = 1 \qquad (1)$$

similarly $\quad 1 = \|g\|_{q_t'} \iff \sum |d_k|^{q_t'} |F_k| = 1 \qquad (2)$

Consider: $\quad \alpha(z) = \dfrac{1-z}{p_0} + \dfrac{z}{p_1} \qquad \alpha(t) = \dfrac{1}{p_t}$

$$\beta(z) = \frac{1-z}{q_0} + \frac{z}{q_1} \qquad \beta(t) = \frac{1}{q_t}$$

$1 - \beta(t) = \dfrac{1}{q_t'}$, $\quad q'$ is the dual power of q.

$$f_z := \sum |c_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{i\theta_j} \chi_{E_j} \qquad c_j = |c_j| e^{i\theta_j}$$

$$g_t := \sum |d_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{i\psi_k} \chi_{F_k} \qquad d_k = |d_k| e^{i\psi_k}$$

On $z = is$

$$\alpha(is) = \frac{1-is}{p_0} + \frac{is}{p_1} = \frac{1}{p_0} + \sqrt{-1}\left(\frac{s}{p_1} - \frac{s}{p_0}\right)$$

$$\beta(is) = \frac{1}{q_0} + \sqrt{-1}\left(\frac{s}{q_1} - \frac{s}{q_0}\right)$$

$$1-\beta(is) = 1 - \frac{1}{q_0} - \sqrt{-1}(\downarrow)$$

Here we used $|c_j|^{\sqrt{-1}\gamma} = e^{\sqrt{-1}\,\ln|c_j|\gamma}$, $\gamma \in \mathbb{R}$ has norm 1

Hence

$$\|f_{is}\|_{p_0}^{p_0 = p(t)} = \sum |c_j|^{\frac{1}{\alpha(t)}} |E_j| = 1, \text{ by } (1)$$

&

$$\|g_{is}\|_{q_0'}^{q_0'} = \sum |d_k|^{\frac{1}{1-\beta(t)} = q'(t)} |F_k| = 1, \text{ by } (2)$$

Similarly $\|f_{1+is}\|_{p_1} = 1$, because

$$\alpha(1+is) = \frac{1-1-is}{p_0} + \frac{1+is}{p_1}$$

$$= \frac{1}{p_1} + \sqrt{-1}\left(\frac{s}{p_1} - \frac{s}{p_0}\right)$$

$$\frac{\alpha(1+is)}{\alpha(t)} = \frac{1}{p_1}\frac{1}{\alpha(t)} = \frac{p_t}{p_1}$$

$$\| f_{(1+is)} \|_{p_1}^{p_1} = 1 \qquad \| g_{(1+is)} \|_{q_1'}^{q_1'} = 1$$

Collecting them we have $\| f_{is} \|_{p_0} = 1 = \| g_{is} \|_{q_0}$

$$\| f_{1+is} \|_{p_1} = 1 = \| g_{1+is} \|_{q_1'}$$

Hence $$\left| \int (T f_{is}) \cdot g_{is} \right| \leq M_0$$

$$\left| \int (T f_{1+is}) \, g_{1+is} \right| \leq M_1$$

By Hörlder inequality & the assumption on

$$T: \; L^{p_0} \longrightarrow L^{q_0} \quad \text{are bounded with } M_0 \; M_1$$

$$L^{p_1} \longrightarrow L^{q_1} \quad \text{as its norm of operator.}$$

Now we consider

$$\phi(z) = \int (T f_z) \, g_z$$

$$= \sum |c_j|^{\frac{\alpha(z)}{\alpha(t)}} |d_k|^{\frac{1-\beta(z)}{1-\beta(t)}} \int (T \chi_{E_j}) \chi_{F_k}$$

holomorphic in z

We may apply 3-line theorem $\Longrightarrow$

$$|\phi(t)| \leq \max_{z=t+is} |\phi(z)| \leqslant M_0^{1-t} M_1^{t}$$

Since $|\phi(is)| \leq M_0$ $\quad |\phi(1+is)| \leqslant M_1$

But $\phi(t) = \int T f_t \; g_t$

$$= \int T f \cdot g$$

Namely $\Rightarrow$ $|\phi(t)| \leqslant M_0^{1-t} M_1^{t}$.

This proves the result.

HY: $\wedge : L^1 \longrightarrow L^\infty$ with $M_0 = 1$ $\quad p_0 = 1 \quad q_0 = \infty$

$L^2 \longrightarrow L^2$ $\quad M_1 = 1 \quad p_1 = 2 \quad q_1 = 2$

$p \in [1\ \ 2]$ $\quad \frac{1}{p_t} = \frac{1-t}{1} + \frac{t}{2} = 1 - \frac{1}{2} t$ $\quad$ Namely $\frac{1}{p} = 1 - \frac{t}{2}$

$$\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{2} = \frac{t}{2} = 1 - \frac{1}{p}$$

Hence: $\wedge : L^p \longrightarrow L^{q = p'}$

satisfies the estimate $\|\hat{f}\|_{L^{p'}} \leqslant \|f\|_{L^p}$ $\quad \forall \quad 1 \leq p \leqslant 2$.