① Uniqueness

Theorem. $f \in L^1$, $\hat{f} = 0$ a.e $\Rightarrow$ $f = 0$ a.e.

Pf. $\underline{n=1 \text{ Case}}$

Let $g(a) = \int_{-\infty}^{a} f(t)\,dt$. It suffices to show $g(a)=0$.

$$\int_{-\infty}^{\infty} e^{-2\pi\sqrt{-1}\,xt} f(t)\,dt = 0 \quad \text{a.e} \quad \Longleftrightarrow \quad \int_{-\infty}^{a} e^{-2\pi\sqrt{-1}\,x(t-a)} f(t)\,dt$$

$$= -\int_{a}^{\infty} e^{-2\pi\sqrt{-1}\,x(t-a)} f(t)\,dt$$

<0

$\operatorname{Re} x + \sqrt{-1}\operatorname{Im} x$

$F_a(x) \doteq \underline{\text{the LHS}}$ $\Rightarrow$ $\forall \operatorname{Im} x > 0$ $F_a(x)$ is welldefined & analytic $\underline{\text{bounded}}$

As a function on $\mathbb{R}$ then extends to $\mathbb{C}$

$\|$

$\underline{\text{RHS}}$

For $\operatorname{Im} x < 0$ it is analytic $\underline{\text{bounded}}$

$|F_a(x)|$ is bounded.

$\Rightarrow$ $F_a(x) \equiv \text{Const}$ by Liouville theorem.

$F_a(0) = g(a)$

$$F_a(\sqrt{-1}s) = \int_{-\infty}^{a} e^{s2\pi(t-a)} f(t)\,dt$$

$e^{s2\pi(t-a)} f(t) \to 0$ as $s \to +\infty$ & $\left|e^{s2\pi(t-a)} f(t)\right| \leq |f(t)|$ $\forall\, t \leq a$ $s>0$.

[Just for fun]. DCT $\Rightarrow$ $F_a(\sqrt{-1}s) \to 0$, as $s \to +\infty$. □

General case follows by view $y_2 \cdots y_n$ $\xi_2 \cdots \xi_n$ as parameters.

② Inversion in L^1

$$f^\vee(x) := \hat{f}(-x) = \int e^{2\pi\sqrt{-1}\langle x, \xi\rangle} f(\xi)\, d\xi$$

Theorem: If $f \in L^1$, $\hat{f} \in L^1 \Rightarrow (\hat{f})^\vee = f$ a.e.

Pf: If $\hat{f} \in L^1$ $(\hat{f})^\vee \in C_0(\mathbb{R}^n)$.

(i) $$\int \hat{f} g = \int f \hat{g}$$

$$LHS = \int \int e^{-2\pi\sqrt{-1}\langle y, \xi\rangle} f(y)\, g(\xi)$$

$$RHS = \int f(y) \int e^{-2\pi\sqrt{-1}\langle \xi, y\rangle} g(\xi)$$

$\hat{f} \in L^\infty$, $\hat{g}$ Hence the integration all makes sense.

(ii) Recall $$\widehat{g^\varepsilon} = \frac{1}{\varepsilon^{\frac{n}{2}}} e^{-\frac{\pi|\xi|^2}{\varepsilon}}$$

$$\Rightarrow \quad \widehat{e^{\sqrt{-1}\, 2\pi\langle \eta, x\rangle} g^\varepsilon} = \frac{1}{\varepsilon^{\frac{n}{2}}} e^{-\frac{\pi|\xi-\eta|^2}{\varepsilon}} = g_\varepsilon^1(\xi-\eta)$$

Hence $$(g_\varepsilon^1 * f)(x) = \int g_\varepsilon^1(x-y) f(y)$$

$$= \int \overbrace{e^{\sqrt{-1}\,2\pi \langle x, \xi \rangle} g^{\varepsilon}(\xi)}^{} \; f(y)\, dy$$

$$\underbrace{\phantom{e^{\sqrt{-1}\,2\pi \langle x, \xi \rangle} g^{\varepsilon}(\xi)}}_{\text{into variable } y \text{ "}}$$

(Namely $\displaystyle\int e^{-2\pi\sqrt{-1}\langle \xi\ y\rangle} \left(e^{2\pi\sqrt{-1}\langle \xi\ x\rangle} g^{\varepsilon}(\xi)\right) d\xi$)

$$\simeq \int e^{2\pi\sqrt{-1}\langle x\ \xi\rangle} g^{\varepsilon}(\xi)\, \hat{f}(\xi)\, d\xi$$

Namely, $\quad (g_\varepsilon' * f)(x) = (g^{\varepsilon} \cdot \hat{f})^{\vee}(x) \qquad (*)$

Taking $\varepsilon \to 0 \Rightarrow \quad \text{LHS} = f(x) \quad$ a.e

$\text{RHS} = (\hat{f})^{\vee}(x).$

$\square$

Same result holds for $f \in L^2$ by approximation

& $(*)$ holds for $f \in L^2$

③ Hausdorff-Young inequality.

Theorem for $1 \le p \le 2$. $\quad f \in L^p \qquad \wedge : f \to \hat{f}$

is a bounded linear operator from $L^p \to L^q$

$\frac{1}{q} + \frac{1}{p} = 1 \qquad$ such that

$$\|\hat{f}\|_{L^q} \le \|f\|_{L^p}$$

We shall use Riesz-Thorin interpolation theorem.

$$T: \quad L^{p_0} \longrightarrow L^{q_0} \qquad\qquad \frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$$

$$L^{p_1} \longrightarrow L^{q_1} \qquad\qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

are bounded with M_0 M_1

$$\|Tf\|_{q_0} \leq M_0 \|f\|_{p_0} \qquad \|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}$$

$\Longrightarrow$

$$\|Tf\|_q \leq M_0^{1-t} M_1^t \|f\|_q .$$

Some convexity in the norm of a linear operator.

The proof needs three-line Lemma.

$\exists$ A generalization by Marcinkiewicz. — involve weak L^p — semi-norm

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HY is a special case of Riesz-Thorin. [See next note]

(4) Polarization formula:

$$(f, f) = (\hat{f}, \hat{f}) \Longrightarrow$$

$$(f, g) = \frac{1}{4}\{(f+g, f+g) + i(f+ig, f+ig) - (f-g, f-g) - i(f-ig, f-ig)\}$$

$$= \frac{1}{4}\{(\hat{f}+\hat{g}, \hat{f}+\hat{g}) + i(\hat{f}+i\hat{g}, \hat{f}+i\hat{g}) - (\hat{f}-\hat{g}, \hat{f}-\hat{g}) - i(\hat{f}-i\hat{g}, \hat{f}-i\hat{g})\}$$

$$= (\hat{f}, \hat{g})$$

Hence if $f_n \in C_c^\infty(\Omega)$ $\|f_n - f\|_{L^2} \to 0$

$\|\hat{f}_n - \hat{f}_m\|_{L^2} \to 0$ as $n, m \to \infty$ $\lim \hat{f}_n$ is defined as $\hat{f}$

Easy to check that it is unique.