① Main Theorem: L^p $1<p<\infty$ is reflexive.

Recall $\wedge: \mathcal{X} \longrightarrow \mathcal{X}^{**}$

$$x \longrightarrow \hat{x}(f) \doteqdot f(x) \quad \forall f \in \mathcal{X}^*$$

$$\|\hat{x}\| \leq \sup \frac{\|f\| \, \|x\|}{\|f\|} = \|x\|$$

& $\exists f_0$ $\|f_0\| = 1$, $f_0(x) = \|x\|$ $\Longrightarrow$ $\|\hat{x}\| \geq \dfrac{|f_0(x)|}{\|f_0\|}$

$= \|x\|$.

Namely $\wedge$: is an isometric embedding.

If $\wedge$: is Onto, we say $\mathcal{X}$ is reflexive.

Stronger statement: (Thm 6.15). $\frac{1}{p} + \frac{1}{q} = 1$

$\forall \phi : (L^p)^*$, $\exists g \in L^q$ such that

$$\phi(f) = \int f g \quad \forall f \in L^p$$

Namely $(L^p)^* \overset{\text{iso}}{\cong} L^q$

as Banach spaces.

Pf: Idea of the proof: Step 1

$\forall g \in L^q$, $\phi_g(f) := \int f g$

& $\|\phi_g\| = \|g\|_q$.

Namely: $g \longrightarrow \phi_g \in (L^p)^*$ is an isometric embedding

Step 2: This is Onto! —— use measure theory

& Radon-Nikodym thm. — A complex version is needed.

Step 1: a) $\|\Phi_g\| \leq \|g\|_q$ — trivial by Hölder

b) Let $f = |g|^{q-1} \operatorname{sgn}(g)$

$\Rightarrow \quad \Phi_g(f) = \int |g|^q$

$$\|f\|_p^p = \left(\int \left(|g|^{q-1}\right)^p\right) = \|g\|_q^q$$

$\frac{1}{p}+\frac{1}{q}=1 \Rightarrow \frac{1}{p} = \frac{q-1}{q}$

$$\Rightarrow \frac{|\Phi_g(f)|}{\|f\|_p} = \frac{\|g\|_q^q}{\|g\|^{q/p}} = \|g\|_q^{q-\frac{q}{p}} = \|g\|_q \quad \square$$

Step 2 takes some effort: It is similar to a more abstract constructions in Ch7, Thm 7.2. Riesz Representation

There, the linear functional is on continuous functions

Here it is on simple functions.

a) $\mu(X) < \infty$.

b) μ is σ-finite

c) general case.

Roughly: A) Φ — acts on simple functions
$\Rightarrow$ a measure ν

B) Radon-Nikodym $\Rightarrow$ $\nu = g\,d\mu$

c) $g \in L^q$

↑ Thm 6.14

② Proof for the case $\mu(X) < \infty$

Define $\qquad \nu(E) = \phi(\chi_E)$.

Clearly $\chi_E \in L^p$ since $\mu(X) < \infty$

If $\quad E = \bigcup E_i$

$\chi_E = \sum \chi_{E_i} \qquad \& \qquad \underbrace{\sum_{i=1}^{N} \chi_{E_i}}_{S_N} \longrightarrow \chi_E$ pointwisely

$S_N \to \chi_E$ also in L^p since

$$|S_N - \chi_E|^p \leq 2^p |\chi_E|^p$$

$$\text{DCT} \Rightarrow \qquad \|S_N - \chi_E\|_{L^p} \to 0 \qquad (*)$$

$$\Rightarrow \sum_{i=1}^{N} \nu(E_i) = \phi\left(\sum_{i=1}^{N} \chi_{E_i}\right) = \phi(S_N)$$

While $\quad \nu(E) = \phi(\chi_E)$

By $\quad \left| \nu(E) - \sum_{i=1}^{N} \nu(E_i) \right| = \left| \phi(S_N - \chi_E) \right| \Rightarrow$

$$\leq \|\phi\| \, \|S_N - \chi_E\|_{L^p} \to 0$$

We may conclude that $\quad \sum_{i=1}^{N} \nu(E_i) \to \nu(E)$

This is enough to conclude $\quad \nu(E) = \sum_{i=1}^{\infty} \nu(E_i)$

(Since order of E_i does NOT matter the convergence is absolute by Baby Rudin)

Recall the Lebsegue-Radon-Nikodym:
p91 of Folland: & Thm 3.12.

$$\frac{d\nu}{d\mu} = g$$

if $\nu \ll \mu$, namely $\nu(E) = 0$ if $\mu(E) = 0$.

[But this is trivial since $\chi_E = 0$ a.e if $\mu(E) = 0$]

So we finished A & B.

Now we show $g \in L^q$:

$\exists \quad \phi_n \longrightarrow g$ a.e $\quad |\phi_1| \leq |\phi_{n+1}| \leq \cdots \leq |g|$, ϕ_n simple

Let $\quad f_n = \dfrac{|\phi_n|^{q-1} \overline{\operatorname{sgn} g}}{\|\phi_n\|_q^{q-1}}$

Observe. (i) $\|f_n\|^p = \displaystyle\int \frac{|\phi_n|^q}{\|\phi_n\|_q^q} = 1$

(ii) $\displaystyle\int |f_n \phi_n| = \int \frac{|\phi_n|^q}{\|\phi_n\|_q^{q-1}} = \|\phi_n\|_q$

Now apply $\phi(f_n) = \displaystyle\int f_n g \, d\mu$

$\Rightarrow \quad \underset{\|\phi\| \cdot 1}{\wedge\!|} \quad = \displaystyle\int \frac{|\phi_n|^{q-1}|g|}{\|\phi_n\|^{q-1}}$

$\Rightarrow \quad \dfrac{1}{\|\phi_n\|_q^{q-1}} \displaystyle\int \underset{|\phi_n|^q}{\underset{\vee\!|}{|\phi_n|^{q-1}|g|}} \leq \|\phi\|$

Taking $n \to \infty$

$\Rightarrow \quad \|\phi_n\|_q \leq \|\phi\|$

Now use $\quad |\phi_n| \to |g| \Rightarrow$

$$\int \underline{\lim} |\phi_n|^q = \int |g|^q$$

$$\leq \underline{\lim}_{n\to\infty} \int |\phi_n|^q \leq \|\phi\| \qquad \text{we have } g \in L^q$$

③ Case μ is σ-finite.

$$X = \bigcup_{n=1}^{\infty} E_n \qquad \mu(E_n) < \infty \qquad E_n \nearrow$$

Consider $\phi : L^p(E_n) \subset L^p(X) \longrightarrow \mathbb{C}$

Then $\exists\ g_n \in L^q(E_n)$ such that $\phi(f) = \int g_n f$
$\forall f \in L^p(E_n)$

Since $\int g_{n+1} f = \int g_n f \quad \forall f \in L^p(E_n)$

$\phi_{g_{n+1}}(f) = \phi_{g_n}(f) \Rightarrow \phi_{g_{n+1}-g_n}(f) = 0 \quad \forall f \in L^p(E_n)$

$\Rightarrow \|g_{n+1} - g_n\|_{L^q(E_n)} = 0 \Rightarrow g_{n+1} = g_n$ a.e on E_n

Now Consider $\lim_{n\to\infty} g_n \doteq g$, a function defined on X.

(Via $g_n = \begin{cases} g_n & \text{on } E_n \\ 0 & \text{outside } E_n \end{cases}$) Since $\int_X |g_n|^q \leq \|\phi\|^q$

$\Rightarrow g \in L^q(X)$. Since $\forall f \in L^p$ $f\chi_{E_n} \in L^p(E_n)$ & $f\chi_{E_n} \to f$ in $L^p(X)$

$$\phi(f) = \lim_{n\to\infty} \phi(f\chi_n) = \lim_{n\to\infty} \int_{E_n} g f = \int_X g f$$

$\square$