① Minkowski ineq

$L^p(X) = L^p(X, \mathcal{M}, \mu)$ $|f|^p$ integrable measurable functions.

$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$

Theorem: $\forall\ f, g \in L^p(X)$ $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

$\infty > p \geq 1$

Pf: Step 1. Young's ineq.

$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ $\frac{1}{p} + \frac{1}{q} = 1$

$\Leftrightarrow$ $a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ $\Leftrightarrow$ $t^{\lambda} \leq \lambda t + 1 - \lambda$

$\left(\frac{a}{b}\right)^{\lambda} \leq \lambda \left(\frac{a}{b}\right) + (1-\lambda)$ $\Leftrightarrow$ $t^{\lambda} - \lambda t + \lambda \leq 1$

$h(t) =$ $h'(t) = \lambda t^{\lambda-1} - \lambda \begin{cases} < 0 & t > 1 \\ > 0 & t < 1 \end{cases}$

$h(t) \leq h(1) = 1$

Step 2. Hölder's inequality.

$$\int |fg| \leq \left(\int |f|^p \right)^{\frac{1}{p}} \cdot \left(\int |g|^q \right)^{\frac{1}{q}}$$

Assume w.L.G $\|f\|_p = \|g\|_q = 1$

$\Leftrightarrow$ $\int |fg| \leq 1$ But $|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}$

$\Rightarrow$ $\int |fg| \leq \frac{1}{p} + \frac{1}{q} = 1$

Step 3. Applying the Hölder $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$

$$\int |f+g|^p = \int |f+g| \cdot |f+g|^{p-1} = \int f \cdot \operatorname{sgn}(f+g) |f+g|^{p-1} + g \cdot \operatorname{sgn}(f+g) \cdot |f+g|^{p-1}$$

$$\leq \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1}$$ $\Rightarrow \square$

② Approximation by simple functions

Thm: $L^p(X)$ is a Banach Space & $\forall f \in L^p$
$\exists f_n$ — simple function in L^p & $f_n \to f$ in L^p-norm.

Pf: Step 1: Abstract statement. $(\mathcal{X}, \|\ \|)$ is complete
$\Leftrightarrow \forall \sum x_n$ if $\sum \|x_n\| < \infty$ $\sum x_n$ is convergent

$(\Leftarrow)$: $\forall \{y_k\}$ Cauchy sequence it sufficent to show
$\exists$ convergent subsequence.

$\forall n \quad \exists k_n \quad \forall k, l \quad k, l \geq k_n$
$\|y_k - y_l\| \leq \frac{1}{2^n}$

Let $x_n = y_{k_n} - y_{k_{n-1}}$

Then $\sum_{n=1}^{\infty} \|x_n\| \leq \|x_1\| + \sum_{n=2}^{\infty} \underbrace{\| y_{k_n} - y_{k_{n-1}} \|}_{\leq \frac{1}{2^{n-1}}} = \|x_1\| + \underbrace{\sum_{n=1}^{\infty} 2^{-n}}_{<+\infty}$

$\Rightarrow \sum_{n=1}^{j} x_n = y_{k_j}$

Hence $\{y_{k_j}\}$ convergent by the assumption.

Step 2: Now if $f_n \in L^p$ $\sum f_n$ has $\sum_{n=1}^{\infty} \|f_n\|_p < +\infty$

$\Rightarrow \bar{S}_N := |f_1| + |f_2| + \dots + |f_N|$ Satisfies $\|\bar{S}_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_p < +\infty$

Namely $\bar{S}_N$ is monotone

$\Rightarrow \underbrace{\sum_{n=1}^{\infty} |f_n|}_{\doteq \bar{S}_\infty}$ exists & $\int \left(\sum_{n=1}^{\infty} |f_n|\right)^p \leq \lim_{N\to\infty} \left[\sum_{n=1}^{N} \left(\int |f_n|^p\right)^{\frac{1}{p}}\right]^p$

$\Rightarrow \underbrace{\{S_N\}}_{= \sum_{i=1}^{N} f_i}$ has a $\underset{\text{Pointwise}}{\underline{\text{limit}}}$ S_∞ $\quad |S_N|, \ |S_\infty| \leq \bar{S}_\infty$

$\Rightarrow S_\infty \in L^p$

Moreover $\quad S_N - S_\infty = \sum_{n=N+1}^{\infty} f_n$

$$\Rightarrow \left(\int |S_N - S_\infty|^p\right)^{1/p} \le \sum_{n=N+1}^{\infty} \|f_n\|_p \longrightarrow 0 \quad \text{as } N \to \infty$$

Namely $\sum f_n$ convergent.

<u>Step 3</u>: $\exists\ f_n$ simple $\quad |f_n| \le |f_{n+1}| \le \dots \ |f|$

$f_n \longrightarrow f$ a.e

$$\Rightarrow \int |f_n|^p < \int |f|^p \qquad \text{Hence } f_n \text{ are all} \in L^p$$

Moreover DCT $\Rightarrow$

$$\lim_{n\to\infty} \int |f_n|^p = \int |f|^p$$

& $\quad |f_n - f|^p \le 2^p |f|^p$

$\Rightarrow \quad \|f_n - f\|_p \longrightarrow 0$ by DCT again.

3 Jensen's ineq & Consequences

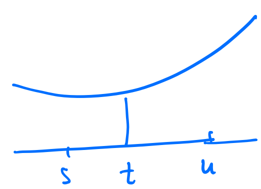
Thm: If μ is a probability measure, i.e $\int_X d\mu = 1$

ϕ is a convex convex function

$$\int \phi(f)\, d\mu \ge \phi\left(\int f\, d\mu\right).$$

<u>Pf</u>:

$$\frac{\phi(t) - \phi(s)}{t-s} \le \frac{\phi(u) - \phi(t)}{u-t} \qquad (1)$$



$$t = \frac{u-t}{u-s}\, s + \frac{t-s}{u-s}\, u$$

$$\Rightarrow \phi(t) \le \frac{u-t}{u-s}\phi(s) + \frac{t-s}{u-s}\phi(u)$$

$$\Leftrightarrow (u-t+t-s)\,\phi(t) \le (u-t)\,\phi(s) + (t-s)\,\phi(u) \Leftrightarrow (1)$$

Now let. $t = \int f \, d\mu$.

$$\beta := \sup \frac{\phi(t) - \phi(s)}{t-s} \quad \forall\, s < t$$

$$\Rightarrow \quad \frac{\phi(t)-\phi(s)}{t-s} \leq \beta_{\ \forall s<t} \quad \& \quad \frac{\phi(s)-\phi(t)}{s-t} \geq \beta \quad \forall\, s > \beta$$

$$\Rightarrow \quad \phi(t) \leq \phi(s) + (t-s)\beta \qquad \text{holds for all } s$$

$$\Rightarrow \quad \phi(t) \leq \phi(f) + (t-f)\beta$$

$$\Rightarrow \quad \phi(t) \int d\mu \leq \int \phi(f)\, d\mu + \beta\left(t - \int f \, d\mu\right)$$

$\Rightarrow$ Jensen's ineq!

Cor: $\log \|f\|_p$ is monotone. in p, provided $\int_X d\mu = 1$

$$\Phi(p) = \frac{1}{p} \log\left(\int |f|^p d\mu\right)$$

$$\Phi'(p) = \frac{1}{p^2} \frac{\int |f|^p \log |f|^p}{\int |f|^p d\mu} - \frac{1}{p^2} \log\left(\int |f|^p d\mu\right)$$

$$\geq 0$$

$$\psi(t) = t \log t \qquad\qquad \psi\left(\int G\right) \geq \int \psi(G)$$

$$\psi'(t) = 1 + \log t$$

$$\psi''(t) = \frac{1}{t}$$

$G = |f|^p$

$$\Rightarrow \left(\int |f|^p\right) \log\left(\int |f|^p\right)$$

$$\geq \int |f|^p \log |f|^p$$

Remark: This holds for all $p > 0$.

(4) Another convexity.

Prop 6.10. $0 < p < q < r \leq \infty$

Then $L^p \cap L^r \subset L^q$ $\qquad \|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$

$$\frac{1}{q} = \lambda\left(\frac{1}{p}\right) + (1-\lambda)\left(\frac{1}{r}\right)$$

Prop 6.10 $\iff$ $\left(\log \|f\|_{(q)}\right) \leq \lambda\left(\log \|f\|_{(p)}\right) + (1-\lambda)\left(\log \|f\|_r\right)$

Namely $\log \|f\|_{(p)}$ is convex in $\left(\frac{1}{p}\right)$

Pf: $\qquad \frac{1}{q} = \lambda\left(\frac{1}{p}\right) + (1-\lambda)\left(\frac{1}{r}\right)$

$$\int |f|^q = \int |f|^{q\lambda} |f|^{(1-\lambda)q}$$

$$\leq \left(\int |f|^p\right)^{\frac{q\lambda}{p}} \left(\int |f|^r\right)^{\frac{(1-\lambda)q}{r}}$$

$p' = \frac{p}{q\lambda}$ $\qquad \frac{1}{p'} + \frac{1}{q'} = 1$ $\qquad$ Hence $\Rightarrow$

$q' = \frac{r}{(1-\lambda)q}$ $\qquad \|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$