① L^1 Case

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi\sqrt{-1}\langle \xi, y\rangle} f(y)\,dy$$

Easy to see

ⓐ $\hat{f}(\xi)$ is continuous, moreover $|\hat{f}(\xi)| \to 0$, as $|\xi| \to \infty$ ← [Riemann-Lebesgue Lemma Exercise]

ⓑ $\widehat{\tau_h f}(\xi) = e^{-2\pi\sqrt{-1}\langle \xi, h\rangle} \hat{f}(\xi)$

$\tau_\eta \hat{f}(\xi) = \hat{h}(\xi) \qquad h(x) = e^{2\pi\sqrt{-1}\langle \eta, x\rangle} f(x).$

ⓒ $\widehat{\delta_\lambda f}(\xi) = \lambda^n \hat{f}(\lambda\xi), \quad (\delta_\lambda f)(x) = f(\frac{x}{\lambda}).$

ⓓ $\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in L^1$

ⓔ If $x^\alpha f \in L^1$, $\forall |\alpha| \le k$ $\quad \hat{f} \in C^k \qquad \partial^\alpha(\hat{f}) = [(-2\pi\sqrt{-1}\, x)^\alpha f]^\wedge$

ⓕ If $f \in C^k$, $\partial^\alpha f \in L^1$, $\forall |\alpha| \le k$ & $\partial^\alpha f \in C_0$ if $|\alpha| \le k-1$

then $\quad (\partial^\alpha f)^\wedge(\xi) = (2\pi\sqrt{-1}\,\xi)^\alpha \hat{f}(\xi)$

Pf. We show ⓒ, ⓔ, ⓕ. [All very simple]

2 A special Case.

<u>Prop 8.24</u>

$$g^\lambda \doteq e^{-\pi\lambda|x|^2}$$

$$\widehat{g^\lambda} = (\lambda)^{-\frac{n}{2}} e^{-\pi|\xi|^2/\lambda}$$

$$= g^1_{\sqrt{\lambda}}(\xi)$$

$\lambda = 0$

$e^{-\pi|x|^2} \longrightarrow e^{-\pi|\xi|^2}$

namely the Gaussian is a fixed point

Better notation is

$G^\lambda \doteq e^{-\pi\lambda^2|x|^2}$

$\widehat{G^\lambda} \doteq G^1_\lambda(\xi)$

Recall $\phi_t(x) = \frac{\phi(\frac{x}{t})}{t^n}$

<u>pf</u>: (i) $\lambda = 1$, is sufficient

(ii) By iterated integrals, reduce it to $n=1$.

(iii) By taking derivative, reduce to ODE

Two slightly different approaches. [FD & LL]

This explored the product structure of $\mathbb{R}^n$ □

3 L^2-Case

[See Thm 8.29 of FD for a slightly different argument]

Theorem: $f \in L^1 \cap L^2$, $\hat{f} \in L^2$

(a) $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ — Plancherel

(b) $\hat{\ }: f \to \hat{f}$ extends uniquely to every $f \in L^2$

(c) $(f, g) = (\hat{f}, \hat{g})$ — Parseval's identity.

$(f, g) := \int_{\mathbb{R}^n} f\bar{g}$

<u>pf</u>:

$$\hat{f}(\xi) = \int e^{-2\pi i\langle\xi, y\rangle} f(y)\,dy$$

We use the approximation by convolution via $e^{-\pi|\xi|^2}$

Consider

$$\int \underbrace{|\hat{f}(\xi)|^2 e^{-\varepsilon\pi|\xi|^2}}$$

Weighted integral of $\hat{f}$

$$= \int \left[\int e^{-2\pi i\langle \xi, x\rangle} f(x)\,dx \cdot \int e^{2\pi i\langle \xi, y\rangle} \bar{f}(y)\,dy \right] \cdot e^{-\varepsilon\pi|\xi|^2}\,d\xi$$

$$= \iiint e^{-\varepsilon\pi|\xi|^2 \underbrace{-2\pi i\langle\xi, x\rangle + 2\pi i\langle \xi, y\rangle}_{= -\pi(2i\langle x-y, \xi\rangle)}} f(x)\bar{f}(y)$$

$$= \iiint e^{-\pi\left(\varepsilon|\xi|^2 + 2\langle i(x-y), \xi\rangle - \frac{1}{\varepsilon}|x-y|^2\right) - \frac{\pi}{\varepsilon}|x-y|^2} f(x)\bar{f}(y)$$

$$= \iiint \underbrace{e^{-\pi\left|\sqrt{\varepsilon}\xi + \frac{i(x-y)}{\sqrt{\varepsilon}}\right|^2}} e^{-\frac{\pi}{\varepsilon}|x-y|^2} f(x)\bar{f}(y)$$

$$\int e^{-\pi\left|\sqrt{\varepsilon}\xi + \frac{i(x-y)}{\sqrt{\varepsilon}}\right|^2} d\xi = 1 \cdot \varepsilon^{-\frac{n}{2}}$$

$$= \iint \left[\frac{1}{\varepsilon^{\frac{n}{2}}} e^{-\frac{\pi}{\varepsilon}|x-y|^2} f(x) \right] \bar{f}(y) \longrightarrow \|f\|_{L^2}$$

$$(g'_{\sqrt{\varepsilon}} * f, f)$$

$$\downarrow$$

$f(y)$ in L^2 as $\varepsilon \to 0$

identity is

$$\boxed{(\hat{f}, \hat{f})_\varepsilon = (f * g'_{\sqrt{\varepsilon}}, f)}$$

weighted

This proves (a) $\Rightarrow$ $\hat{f} \in L^2$ $\square$

④ Added remarks.

ⓐ $$\int e^{-\pi|\eta+\sqrt{-1}z|^2} = \int e^{-\pi|\eta|^2} \qquad \eta, z \in \mathbb{R}^n$$

Express it as $\prod_{j=1}^{n} \int e^{-\pi(\eta_j+\sqrt{-1}z_j)^2} d\eta_j$

We reduce it to $n=1$ case

$$\int e^{-\pi(\eta+\sqrt{-1}z)^2} = \int e^{-\pi\eta^2}$$

$$F(z) := \int e^{-\pi(\eta+\sqrt{-1}z)^2} d\eta$$

$$F'(z) := \int e^{-\pi(\eta+\sqrt{-1}z)^2} \cdot -2\pi(\eta+\sqrt{-1}z)\, d\eta$$

$$= \int d\, e^{-\pi(\eta+\sqrt{-1}z)^2} = e^{-\pi(\eta+\sqrt{-1}z)^2}\Big|_{\eta=-\infty}^{\eta=+\infty} = 0$$

ⓑ $f*\phi_t \to f$ uniformly on a compact subset $K \subset U$

On U, f is bounded & continuous

$$(f*\phi_t - f)(x) = \int (f(x-y)-f(x)) \frac{\phi(\frac{y}{t})}{t^n} dy$$

$$= \int [f(x-tz)-f(x)]\, \phi(z)\, dz$$

Hence $|f*\phi_t - f|(x) \leq \int |f(x-tz)-f(x)|\, |\phi(z)|\, dz$

$\|f*\phi_t - f\|_{L^\infty(K)} \leq \int \underbrace{\sup_{x\in K} |f(x-tz)-f(x)|}_{G(t,\ z)}\, |\phi(z)| \longrightarrow 0$ as $t\to 0$ by DCT.

$\forall z.\ G(t, z) \to 0$ & $\leq 2\|f\|_{L^\infty(U)}$ for $t \ll 1$