This is NOT the easy version of Analysis!

Hardwork is expected!

Homeworks are crucial! Do them.

Topics covered:

covered by qualifying exam.

- Fourier transform.
- Distributions
- Some further topics on L^p-spaces
- Riesz-Markov Representation & Radon measures
- An intruction to BV-functions.

1 Notations

(i) $\alpha = (\alpha_1, \dots \alpha_n)$ $\alpha_i \in \mathbb{Z}$. $\alpha_i \geq 0$

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$$

$$\alpha! = (\alpha_1)! \cdots (\alpha_n)!$$

$$\partial^\alpha (f g) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} \partial^\beta f\, \partial^\gamma g$$

Ex: Prove it

(ii) $(\tau_y f)(x) = f(x-y)$.

f is defined on $\Omega \subset \mathbb{R}^n \Longrightarrow (\tau_y f)$ is defined on $\Omega + y$

(iii) $$f * g\,(x) \doteq \int_{\mathbb{R}^n} f(\underbrace{x-y}_{\substack{\parallel \\ z}})\, g(y)\, dy = \int_{\mathbb{R}^n} f(\underset{z}{y})\, g(\underset{x-z}{x-y})\, \underset{dz}{dy}$$

(iv). Generalized Minkowski inequality: $p \geq 0$

$$\left[\int \left(\int f(x,y)\, d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{p}} \leq \int \left[\int f^p(x,y)\, d\mu(x)\right]^{\frac{1}{p}} d\nu(y).$$

② Approximation via convolutions.

Let $\phi \in L^1$ $\quad \phi_t(x) := \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$

$$\int \phi_t(x) = \int \frac{1}{t^n}\phi(\underbrace{x/t}_{y})\, dx = \int \phi(y)\, dy$$

$dy = dx/t^n$

W.L.G $\int \phi(y)\, dy = 1.$

This is the case we use the approximation techs.

[Observe: If $\phi \in C_c^\infty(\mathbb{R}^n)$ — compact supported in, say $B(0,1)$ — the supp ϕ_t shrinks $\xrightarrow{\text{as } t \to 0}$ a point $0 \in \mathbb{R}^n$.]

See (8.1)-eqn & Exercise 3 of P239.

Theorem 8.14: (i) $f \in L^p \Rightarrow f * \phi_t \to f$ in L^p as $t \to 0$

(ii) f is bounded & uniformly continuous

$\Rightarrow f * \phi_t \longrightarrow f$ uniformly.

Pf: Easy.

Theorem 8.15 If $|\phi(x)| \le \frac{C}{(1+|x|)^{n+\varepsilon}}$ $\varepsilon>0$, & $f \in L^p$, $1\le p\le\infty$

then $f*\phi_t \to f$ on x, with x being an Lebesgue point of f.

Pf: Lebesgue point $x \iff$

$$\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)-f(x)|\,dy = 0$$

Thm 3.21

(i) Easy case: ϕ is as (8.1). Namely $\operatorname{supp}\phi \subset B(0,1)$

$$\Rightarrow \quad f*\phi_t(x) - f(x)$$

$$= \int f(x-y)\,\phi(y/t)\,\frac{1}{t^n}\,dy - \int \left[\phi\left(\frac{y}{t}\right)\frac{1}{t^n}\right] f(x)$$

$$= \int \left[f(x-y) - f(x)\right]\phi\left(\underbrace{\frac{y}{t}}_{z}\right)\frac{1}{t^n}\,dy$$

$$\left(= \int \left[f(x-tz) - f(x)\right]\phi(z)\,dz \right)$$

$$\Rightarrow \quad \le \frac{1}{t^n}\int_{B(0,t)} |f(x-y) - f(x)|\; C\,dy$$

$$= \frac{C}{t^n}\int_{B(x,t)} |f(z) - f(x)|\,dz \longrightarrow 0$$ since x is a Lebesgue point.

Here we only use $f \in L^1_{loc}$.

(ii) General Case: ϕ does not have compact support.

But $|\phi(x)| \lesssim \dfrac{C}{(1+|x|)^{n+\varepsilon}}$

Namely $|\phi(x)| \leq C$ (E1) & for $|x|$ large it decays to 0 faster than $\dfrac{1}{|x|^n}$.

In particular $|x| \geq 1 \Rightarrow$

$$|\phi(x)| \lesssim \frac{C}{|x|^{n+\varepsilon}} \qquad \text{(E2)}$$

$\forall\, \delta > 0\ \exists\, \eta$ such that if $r \leq \eta$

$$\int_{B(0,r)} |f(x-y) - f(x)|\, dy \leq \delta\, r^n \qquad \text{(E3)}$$

Now

$$|f * \phi_t(x) - f(x)| = \left| \int [f(x-y) - f(x)]\, \phi_t(y)\, dy \right|$$

$$\leq \int |f(x-y) - f(x)|\, |\phi_t(y)|\, dy$$

$$= \underbrace{\int_{|y|\geq\eta} |f(x-y) - f(x)|\, |\phi_t(y)|}_{A''} + \underbrace{\int_{|y|<\eta} \boxed{}}_{B=}$$

We say estimate each of them. ($1 < p < \infty$. the other is the same)

$$A \leq \frac{1}{t^n}\left[\int_{|y|\geq\eta} |f(x-y)| \cdot \left|\phi\left(\frac{y}{t}\right)\right| + \int_{|y|\geq\eta} |f(x)|\, \left|\phi\left(\frac{y}{t}\right)\right| \right]$$

$$\leq \frac{1}{t^n}\left[\|f\|_{L^p} \left(\int_{|y|\geq\eta} \left|\phi\left(\frac{y}{t}\right)\right|^q \right)^{\frac{1}{q}} + |f(x)| \int_{|y|\geq\eta} \left|\phi\left(\frac{y}{t}\right)\right| \right]$$

$$\frac{1}{t^n}\left(\int_{|y|\geq\eta} \left|\phi\left(\frac{y}{t}\right)\right|^q dy \right)^{\frac{1}{q}} \underset{\uparrow \text{E2}}{\lesssim} \frac{1}{t^n}\, C' \left(\int_\eta^\infty \left|\frac{y}{t}\right|^{-q(n+\varepsilon)} \underbrace{|y|^{n-1}\, d|y|}_{\text{Polar coordinate}} \right)^{\frac{1}{q}}$$

$$\leq C' t^{(n+\varepsilon)-n} \boxed{\left(\int_\eta^\infty \underbrace{r^{-q(n+\varepsilon)} r^{n-1}}\, dr\right)^{1/q}} = C(\eta) < \infty$$

$\searrow 0$ as $t \to 0$ since $q \geq 1$

Similarly $\displaystyle \frac{1}{t^n} \int_{|y|\geq\eta} \left|\phi\left(\frac{y}{t}\right)\right| \leq C' t^{n+\varepsilon-n} \boxed{\int_\eta^\infty |y|^{-n-\varepsilon} \underbrace{|y|^{n-1}}\, d|y|}$ $= C(\eta) < \infty$

E2

$\longrightarrow 0$ as $t \to 0$ $\quad A \longrightarrow 0$ as $t \to 0$ E4

For $\displaystyle B = \int_{|y|<\eta} |f(x-y) - f(x)|\, |\phi_t(y)|\, dy$

decompose it into annulus. use $K \in \mathbb{Z}$, $2^K \leq \frac{\eta}{t} < 2^{K+1}$

Namely $\left|\frac{y}{t}\right| < \frac{\eta}{t} < 2^{K+1}$. $\quad \underline{2^{-j}\eta \leq |y| < 2^{-j+1}\eta}$

on the j-th annulus $\quad 2^{-j}\left(\frac{\eta}{t}\right) \leq \left|\frac{y}{t}\right| < 2^{-j+1}\left(\frac{\eta}{t}\right)$
$\qquad \geq 2^K \qquad 2^K$

$\Rightarrow$ Hence $\quad 1 \leq \left|\frac{y}{t}\right| \leq 2^{-(j-1)}\left(\frac{\eta}{t}\right)$

Use the estimate

$$\int_{|y|\leq\eta} \square = \underbrace{\sum_{j=1}^K \int_{2^{-j}\eta \leq |y| < 2^{-j+1}\eta} \square}_{= B_1}$$

$j = 1, \dots K$

$$+ \underbrace{\int_{|y|< 2^{-K}\eta} \square}_{B_2} \leq \frac{C\delta(2^{-K}\eta)^n}{t^n} \leq C\delta 2^n$$

E5

radius $\leq (2^{-K}\eta)$ E3

$$\int_{2^{-j}\eta \le |y| \le 2^{-j+1}\eta} |f(x-y)-f(x)|\,\left|\phi_t\left(\frac{y}{t}\right)\right| \underset{E2}{\le} \frac{1}{t^n}\int_{2^{-j}\eta \le |y| \le 2^{-j+1}\eta} |f(x-y)-f(x)| \frac{C}{\left|\frac{y}{t}\right|^{n+\varepsilon}}$$

$$\le C t^{\varepsilon}\left(2^{-j}\eta\right)^{-(n+\varepsilon)} \int_{|y|\le 2^{-j+1}\eta} |f(x-y)-f(x)|$$

$$\frac{1}{|y|^{n+\varepsilon}} \le \left(2^{-j}\eta\right)^{-(n+\varepsilon)}$$

$$\left(\le \delta\left(2^{-j+1}\eta\right)^{n}\right)$$

$$\underset{E3}{\le} C\delta\left(\frac{t}{\eta}\right)^{\varepsilon} 2^{+j(n+\varepsilon)}\, 2^{-jn}\cdot 2^{n} \qquad \leftarrow \quad E6$$

$$\le \underbrace{C'(n)}_{2^n \to} \delta\left(\frac{t}{\eta}\right)^{\varepsilon} 2^{j\varepsilon}$$

Hence

$$B_1 \le \left(\sum_{j=1}^{K} 2^{j\varepsilon}\right) C'(n)\,\delta\left(\frac{t}{\eta}\right)^{\varepsilon}$$

$$\rotatebox{90}{$\le$} \quad \frac{2^{(K+1)\varepsilon}-2^{\varepsilon}}{2^{\varepsilon}-1} \le \frac{2^{K\varepsilon}\cdot 2^{\varepsilon}}{2^{\varepsilon}-1}$$

$$\le C''(n,\varepsilon)\,\delta\left(\frac{t}{\eta}\right)^{\varepsilon}\cdot\left(2^{K}\right)^{\varepsilon}$$

$$\frac{\eta}{t} \ge 2^{K} \quad \Rightarrow \left(\frac{t}{\eta}\right) \le 2^{-K}$$

$$\le C''(n,\varepsilon)\,\delta \qquad E7$$

$$B \le \tilde{C}(n,\varepsilon)\,\delta. \qquad \square$$

Conclusion: L^p function can be approximated by smooth ones with compact supports!

We need this to define Fourier transforms on L^2

Also useful in the discussion of distributions.