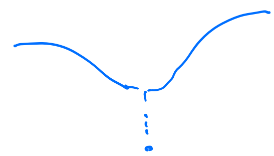
① LSC & USC functions.

LSC $:= \{ f \mid$ such that $\{f > a\}$ is open $\forall a \}$

$f(x) \in (-\infty, +\infty]$



USC $:= \{ f \mid \{f < a\}$ is open$\}$ $\quad f(x) \in [-\infty, +\infty)$.

Clearly if $f \in$ LSC $\Rightarrow -f \in$ USC

$\left(-f > -a \iff f < a \right)$

Basic Properties: a) U is open $\chi_U \in$ LSC

b) If $f \in$ LSC $\Rightarrow cf \in$ LSC if $c \geq 0$

c) If $\mathcal{G}$ is a family of LSC $\quad f(x) = \sup\{ g(x),\ g \in \mathcal{G} \}$

$\Rightarrow f$ is LSC.

d) f_1 & f_2 are LCS $\Rightarrow f_1 + f_2 \in$ LSC.

e) If X is LCH, $\quad f \geq 0 \quad f \in$ LSC

$\Rightarrow f(x) = \sup\{ g(x),\ g \in C_c(X)\ 0 \leq g \leq f \}$.

Pf: a), b) easy

c) $f(x_0) > a \Rightarrow \exists g \quad g(x_0) > a$

$\Rightarrow \exists U \ni x_0 \quad g(x) > a \quad \forall x \in U$

$\Rightarrow f(x) > a$ in $U \Rightarrow \tilde{U} = \{ f(x) > a \}$ is open

d) $(f_1 + f_2)(x_0) > a \Rightarrow f_1(x_0) > a - f_2(x_0) + \varepsilon$ for some $\varepsilon > 0$

$\Rightarrow f_1(x) > a - f_2(x_0) + \varepsilon$ for $x \in U_1 \ni x_0$

& $f_2(x) > f_2(x_0) - \varepsilon$ for $x \in U_2 \ni x_0$

$\Rightarrow \quad f_1(x) + f_2(x) > a \quad$ if $\quad x \in U_1 \cap U_2$

Namely $\quad \{ (f_1 + f_2) > a \}$ is open

e). $\forall x$ if $f(x) > 0$. Consider $0 < a < f(x)$.

$U := \{ f(y) > a \}$ is a open neighborhood of x

$\Rightarrow \exists\, g \in C_c(X) \quad g(x) = a \quad \& \quad 0 \le g \le a \chi_U < f$

$K = \{x\} \subset U \quad \exists\, g_1$

$g_1(x) = 1, \quad \operatorname{supp} g_1 \subset U, \quad 0 \le g_1 \le 1$

Hence $\quad \sup_{g \in C_c(X)} \{ g(x),\ 0 \le g \le f \} \geq a$

$\square$

(2) Approximations (the integral of a LSC function by $C_c(X)$ ones)

<u>Prop 7.12</u>. Let $\mathcal{G}$ be a family of LSC functions ≥ 0 on a LCH space X.

Assume $\mathcal{G}$ is a directed set by $\le$, Namely $\forall\, g_1, g_2\ \exists\, g$
$g_1 \le g,\ g_2 \le g$

Let $f = \sup\{ g : g \in \mathcal{G} \}$. If μ is any Radon measure

then $\quad \int f \, d\mu = \sup \left\{ \int g \, d\mu.\ g \in \mathcal{G} \right\}$.

<u>Corollary 7.13</u>: If μ is Radon & $f \geq 0$ $f \in LSC$.

then $\quad \int f \, d\mu = \sup \left\{ \int g \, d\mu.\ g \in C_c(X) \quad 0 \le g \le f \right\}$

$\mathcal{G} := \{ g \in C_c(X).\ 0 \le g \le f \} \quad \Big\} \Rightarrow$ Corollary

Clearly if $g_i \in \mathcal{G} \qquad \underset{\tilde{g}}{\underbrace{\max(g_1, g_2)}} \in \mathcal{G}.$

$\tilde{g} \geq g_1 \quad \tilde{g} \geq g_2$

<u>Pf of the Proposition</u>: We need to use the Construction in 2.10.

$$\phi_n = \frac{1}{2^n}\sum_{k=0}^{2^{2n}-1} k\,\chi_{E_n^k} + 2^n\chi_{F_n}$$

$$E_n^k := \left\{x \;\middle|\; \frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}\right\} \qquad F_n = \{x \mid f > 2^n\}$$

$$0 < f - \phi_n < \frac{1}{2^n} \quad \text{on} \quad F_n^c$$

We rewrite it as

$$\phi_n = \frac{1}{2^n}\sum_{j=1}^{2^{2n}} \chi_{U_n^j} \qquad U_n^j := \left\{x \;\middle|\; f(x) > \frac{j}{2^n}\right\}$$

$$F_n \in U_n^j \quad \forall\ 1 \leq j \leq (2^n)^2$$

$$E_n^k \in U_n^j \quad \text{for } j \leq k \qquad \text{Hence} \qquad k\,\chi_{E_n^k}$$

$$\phi_n \nearrow f \Rightarrow \forall\, a < \int f$$

$$\Rightarrow \exists\, n \gg 1 \quad \int \phi_n > a \quad \Leftrightarrow \quad \frac{1}{2^n}\sum_{j=1}^{(2^n)^2}\int \chi_{U_n^j} > a$$

Now for U_n^j open $\exists$ K_n^j such that $K_n^j \subset U_n^j$

$$\frac{1}{2^n}\sum_{j=1}^{(2^n)^2}\int \chi_{K_n^j} > a$$

Now let $\psi := \frac{1}{2^n}\sum_{j=1}^{(2^n)^2} \chi_{K_n^j} \leq \phi_n$

Now we construct open cover of $\bigcup K_n^j$

$\forall x \in \bigcup K_n^j$ by $f(x) = \sup\{ g(x) \mid g \in \mathcal{G} \}$

$\Rightarrow \exists\ g_x$ such that $g_x(x) > \psi(x)$.

Consider $V_x := \{ g_x(y) - \psi(y) > 0 \}$ which is open & $x \in V_x$

$\bigcup_{x \in \bigcup K_n^j} V_x$ covers $\bigcup K_n^j \Rightarrow \exists$ finite cover

$$\bigcup_{\ell=1}^{m} V_{x_\ell} \supset \bigcup K_n^j$$

$\Rightarrow \exists\ g \in \mathcal{G}$ such that $g \geq g_{x_j}$

$\Rightarrow \psi \leq g \Rightarrow \int g \geq \int \psi > a$

Since clearly

$$\int f \geq \int g \quad \forall g \in \mathcal{G}$$

$$\Rightarrow \int f = \sup_{g \in \mathcal{G}} \left\{ \int g \right\}.$$

$f = \chi_U$

$\mu(U) = \sup\{ \int g : g \in C_c(X),\ 0 \leq g \leq \chi_U \}$

③ $\int f = \inf \left\{ \int g\, d\mu,\ g \geq f,\ g \text{ is LSC} \right\}$

Prop 7.14.

$f = \chi_E$

$\mu(E) = \inf\{ \mu(U),\ U \supset E \}$
$\chi_U \geq \chi_E$

proof is relatively easier.