① Borel regular on $\mathbb{R}^n$.

Thm 3.22 ν is a signed or complex measure on $\mathbb{R}^n$

$|\nu|$ is regular. (i.e. $|\nu|(K) < \infty$. $\forall$ K compact).

$d\nu = d\lambda + f\,dm$ be its Lebesgue–Radon–Nikodym decomposition.

Then $\forall$ family $\{E_r\}$ shrinks to x nicely.

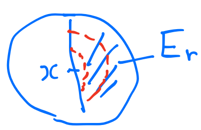
$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x). \quad \underline{a.e}.\ x \in \mathbb{R}^n$$

Pf: $d\nu = d\lambda + f\,dm$ is just $\underbrace{d\nu,\ dm}$ Application of Thm 3.8, Thm 3.12.

$\{E_r\}$ SNT x if $E_r \subset B(x,r)$

$|E_r| \geq \delta\, |B(x,r)|$ for some $\delta > 0$.

E.g



Observation: $\lim_{r\to 0} \displaystyle\fint_{E_r} |f(y) - f(x)|\, dy \longrightarrow 0$ a.e

Since

$$\fint_{E_r} |f(y) - f(x)|\, dy \leq \frac{|B(x,r)|}{|E_r|} \fint_{B(x,r)} |f(y) - f(x)|\, dy$$

$$\leq \frac{1}{\delta} \fint_{B(x,r)} |f(y) - f(x)|\, dy \to 0$$

To prove the result, it suffices to show

$$\frac{|\lambda(E_r)|}{|E_r|} \to 0 \quad \text{as } r \to 0. \quad \text{a.e } x \in \mathbb{R}^n.$$

$\lambda \perp m$.

By the fact $\lambda \perp m$ $\exists$ A $\lambda(A)=0$. & $\mu(A^c)=0$
on Borel algebra

We only need to show $\forall$ $x \in A$

$$\lim_{r\to 0} \frac{\lambda(B_r)}{|B_r|} \to 0 \quad \text{as } r \to 0 \quad \text{a.e } x \in A$$

Let $F_k := \{ x \in A \mid \overline{\lim_{r\to 0}} \frac{\lambda(B_r)}{|B_r|} > \frac{1}{k} \}$

$\forall x \in F_k$ $\exists$ $B_{r_x}(x)$ $\lambda(B_{r_x}) > \frac{1}{k} |B_{r_x}|$

Now we consider the covering of F_k by B_{r_x}

Let $V := \bigcup_{x \in F_k} B_{r_x}$

By the covering Lemma $\exists$ $B_{r_{x_j}}$

$\forall c < m(V)$

$$c < 3^n \sum_{j=1}^{\ell} |B_{r_{x_j}}| \leq 3^n k \sum_{j=1}^{\ell} \lambda(B_{r_{x_j}})$$

$B_{r_{x_j}}$ are disjoint

On the other hand $\lambda(A)=0 \Rightarrow \forall \varepsilon > 0$, $\exists$ $U_\varepsilon \supset A$

$\lambda(U_\varepsilon) < \varepsilon$

Now we may choose $B_{r_x} \subset U$ in the above step

$\Rightarrow$ $c < 3^n k \varepsilon$ $\Rightarrow$ $0 = m(V) \geq m(F_k)$. $\square$

② X LCH, μ-Radon, $C_c(X)$ is dense in $L^p(X, \mu)$ $1 \le p < \infty$.

Pf: $\forall f \in L^p(X, \mu)$.

$\exists \ \varphi_n$ simple $|\varphi_n| \le |\varphi_{n+1}| \le \cdots \le |f|$ $\varphi_n \to f$

& $\lim_{n\to\infty} \int |\varphi_n|^p = \int |f|^p$

Moreover $|f - \varphi_n|^p \le 2^p(|f|^p + |\varphi_n|^p)$

$\Rightarrow \lim_{n\to\infty} \int |f-\varphi_n|^p = 0$

Namely simple functions are dense in $L^p(X, \mu)$.

$\varphi = \sum_{i=1}^{\ell} a_i \chi_{E_i}$ E_i disjoint

We now approximate χ_{E_i} by $C_c(X)$.

$\forall E$. $\mu(E) < \infty$. $\exists \ U \supset E$ $\mu(U) < \mu(E) + \varepsilon$.

Since by 7.5. $\exists K \subset E$ $\mu(K) \ge \mu(E) - \varepsilon$

Then consider $K \prec f \prec U$

$\Rightarrow \int |f - \chi_E| \le \int_{U \setminus K} |f - \chi_E| \le 2 \times 2\varepsilon = 4\varepsilon$

& $\int |f - \chi_E|^p \le 2^p \cdot 2\varepsilon$ as well.

$\Rightarrow \|f - \chi_E\|_p \longrightarrow 0$

③ Lusin's theorem: f is measurable w.r.t μ - a Radon measure.

X LCH

Assume that $\mu(\{|f| \neq 0\}) < +\infty$. Then $\forall \varepsilon > 0$

$\exists \phi \in C_c(X)$ such that $\mu(\{\phi \neq f\}) < \varepsilon$

& If f is bounded ϕ can be chosen such that

$\|\phi\|_u \leq \|f\|_u \rightsquigarrow$ namely $\sup_{x \in X} |f(x)|$

W.L.G assume f is bounded

Pf: Since $f \in L^1 \Rightarrow \exists g_n \to f$ in L^1, $g_n \in C_c(X)$.

Then $\{g_{n_j}\}$ still write as $g_n \to f$ a.e

$\Rightarrow \exists A \subset E := \{x \mid |f(x)| \neq 0\}$

Such that $g_n \rightrightarrows f$ on A & $\mu(E \setminus A) < \varepsilon/3$

Now find (i) $K \subset A$ compact $\mu(A \setminus K) < \varepsilon/3$

(ii) $U \supset E$ open $\mu(U \setminus E) < \varepsilon$

$\Rightarrow$ f is continuous on K.

Now we use Tietze extension thm: f can be extended to a function h

X such that $h = 0$ outside U. $h \in C_c(X)$

$h = f$ on K.

To get the desired estimate one simply need to "chop" h.

$$\beta(z) = \begin{cases} z & |z| \leq \|f\|_u \\ \|f\|_u \cdot \dfrac{z}{|z|} & \text{if } |z| > \|f\|_u \end{cases}$$

$\phi = \beta(h)$ Satisfies $\|\phi\|_u \leq \|f\|_u$

$\mu(\{x \mid \phi \neq f\}) \leq \varepsilon$.