1 Hahn-Banach Thm for linear spaces over Complexs

$\mathcal{X}$. Complex vector space.

$p(x)$: is a semi-norm: Namely $p: \mathcal{X} \to [0, \infty)$

(i) $p(x+y) \leq p(x) + p(y)$

(ii) $p(\lambda x) = |\lambda| \, p(x)$.

Theorem: If f is a complex linear functional on $\mathcal{M} \subset \mathcal{X}$

i.e $f: \mathcal{X} \to \mathbb{C}$ $\quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$

$\mathcal{M}$ being a Complex vector subspace.

Assume that $|f(x)| \leq p(x)$ $\quad \forall x \in \mathcal{M}$

Then $\exists$ an extension $F(x)$ of $f(x)$ satisfies

$$f(x) = F(x) \quad \forall x \in \mathcal{M}$$

$$\& \quad |F(x)| \leq p(x).$$

Remark: The assumption is slightly stronger.

Pf: $f(x) = u(x) + i v(x)$.

But f is linear $\quad f(ix) = i f(x) = i u(x) + (-1) v(x)$

$\qquad \qquad \qquad \qquad \| $

$\qquad \qquad u(ix) + i v(ix) \quad \Longrightarrow \quad v(x) = -u(ix)$

Namely $\quad f(x) = u(x) - i u(ix)$.

By the assumption $\quad |u(x)| \leq |f(x)| \leq p(x)$.

$\Rightarrow$ The real HBT applies $\Rightarrow$ $\exists$ U such that

$U(x+y) = U(x) + U(y)$ & $U(x)\big|_{\mathcal{M}} = u(x)$

$U(\lambda x) = \lambda U(x) \quad \forall \lambda \in \mathbb{R}$

Let $F(x) = U(x) + (-i)\, U(ix)$

$ix \in \mathcal{M}$ if $\mathcal{M} \subset X$ is a complex-linear sub-space

Then $F(x)\big|_{\mathcal{M}} = u(x) + (-i)\, u(ix) = f(x)$

$$F(ix) = U(ix) - i\,U(-x) = i\left[U(x) - i\,U(ix)\right] = i\,F(x)$$

Namely F is i-linear as well.

[$F(x+y) = U(x+y) + (-i)\, U(ix+iy) = F(x) + F(y)$

$F(\lambda x) = \lambda\left[U(x) - i\,U(ix)\right]$]

Now $|F(x)| = e^{-i\theta} F(x) = F(e^{-i\theta} x) = U(e^{-i\theta} x)$

$\theta = \arg(F(x))$ $\leq p(e^{-i\theta} x) = p(x)$.

[Here is where we used $p(\lambda x) = |\lambda|\, p(x)$].

(2) Hyperplane Seperation theorem for Convex set.

<u>Thm</u>: Let K be a nonempty convex subset such that every $x \in K$ is an interior (in the sense of Lax). Let $y \bar{\in} K$

Then $\exists\, \ell$, linear functional: $\ell(x) < c$, $\forall x \in K$.

$\ell(y) = c$

Def: $x \in K$, is interior, iff $\forall y \in \mathcal{X}$ $\exists \varepsilon_y$ s.t. $\forall |t| < \varepsilon_y$, $x + t y \in K$

[This is a finer topology]

Defn: Assume $0 \in K$ is an interior point.

$$P_K(x) \doteq \inf a, \quad a > 0, \quad \frac{x}{a} \in K.$$

$P_K(x)$ is called the gauge function

Pf: Key Facts:

(a) $P_K(x)$ is homogeneous of degree 1

(b) $P_K(x+y) \leq P_K(x) + P_K(y)$

(c) $x \in K \Rightarrow P_K(x) \leq 1$, & x is interior iff $P_K(x) < 1$

Now let ℓ: $\langle y \rangle$ with $\ell(y) = 1$ $\quad \ell(\lambda y) = \lambda$

Then $\ell(\lambda y) \leq P_K(\lambda y)$, since $\ell(y) = 1$ $\quad P_K(y) \geq 1$

$\ell(\lambda y)$ for $\lambda \leq 0 \Rightarrow \ell(\lambda y) \leq 0$ & $P_K(\lambda y) \geq 0$

$\forall \lambda > 0$ $\quad \ell(\lambda y) = \lambda$ $\quad P_K(\lambda y) = \lambda P_K(y) \geq \lambda$

By Hahn-Banach, $\Rightarrow$ ℓ can be extended to $\mathcal{X}$ with

$$\ell(x) \leq P_K(x)$$

Hence $\ell(x) \leq P_K(x) < 1$ $\quad \forall x \in K$. $\square$

Now we prove (a) – (c) for the gauge function

3 Details on the gauge function

$\forall\, y \quad \exists\, \varepsilon_y \qquad ty \in K \quad \text{if} \quad |t| < \varepsilon_y$

Hence $\quad P_K(y) < \infty$

$P_K(x+y) \leqslant P_K(x) + P_K(y)$.

By definition, $\forall \varepsilon > 0$. $\quad \frac{x}{a} \in K \quad \text{if} \quad a \leqslant P_K(x) + \varepsilon/2$

$\& \quad \frac{y}{b} \in K \quad \text{if} \quad b \leqslant P_K(y) + \frac{\varepsilon}{2}$

Then $\frac{a}{a+b} \cdot \frac{x}{a} + \frac{b}{a+b} \frac{y}{b} = \frac{x+y}{a+b} \quad \in K$

$\Rightarrow \qquad a+b \geqslant P_K(x+y)$

$\Rightarrow \quad P_K(x+y) \leqslant P_K(x) + P_K(y) + \varepsilon \qquad \forall \varepsilon > 0$

We have (b)

(a) is obvious since $\frac{\lambda y}{\lambda a} \in K$ iff $\frac{y}{a} \in K$

since $\frac{x}{1} \in K$

(c) $\quad x \in K \Rightarrow \quad P_K(x) \leqslant 1$

If x is an interior point $\Rightarrow \quad x + tx \in K$ for $|t| < \varepsilon$

Then $\quad (1+t)x \in K \Rightarrow \quad \frac{x}{1-\delta} \in K \quad$ with $\frac{1}{1-\delta} = 1+t$

$\Rightarrow \quad P_K(x) \leqslant 1-\delta \quad < 1.$

If $P_K(x) < 1 \quad \Rightarrow \exists\, \varepsilon \qquad P_K(x) \leqslant 1-\varepsilon$, $\quad$ Pick $\frac{x}{1-\varepsilon'} \in K$ $\quad 1-\varepsilon' > 1-\varepsilon$

Hence $\forall y \in \mathcal{X}$. $\quad \exists\, \varepsilon_y \qquad ty \in K$ if $|t| < \varepsilon_y$

$x + \bar{t} y = (1-\varepsilon')\boxed{\left(\frac{x}{1-\varepsilon'}\right)} + \underset{\shortparallel}{\varepsilon'}\left(\underbrace{\left(\frac{\bar{t}}{\varepsilon_1}\right)}_{<\varepsilon_y} y\right) \quad \in K \qquad \frac{\bar{t}}{\varepsilon_1} < \varepsilon_y$ is enough.